**Math 028b April 2002 Final – Solutions**
March/02 test

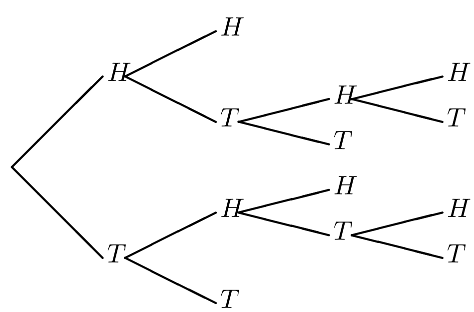
**Question 1**

We can partition event $E$ into those elements of $E$ which are also in event $F$ and those which are not. We have

$$\begin{aligned} n(E) &= n(E \cap F) + n(E \cap F^C) \\ \Rightarrow n(E \cap F) &= n(E) - n(E \cap F^C) = 40 - 30 = 10 \end{aligned}$$

**Question 2**

We draw a counting tree. There are branchings for every time that a coin might be tossed. A path ends if there have been 2 Heads in a row, 2 Tails in a row or if 4 tosses have been made. Otherwise, the path continues with a new branching. We have:




Counting terminal points, we see that there are 8 sequences of tosses possible.

**Question 3**

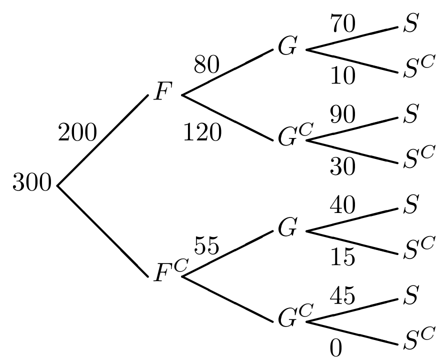
We start by filling in a couple of numbers on the tree. We know that $n(U) = 300$ and $n(F) = 200$, so we must have $n(F^C) = 300-200 = 100$. That is, 100 tourists did not visit France.

Of these 100, there were 40 who visited both Germany and Spain, and 45 who visited Spain but not Germany, while there was nobody who didn't visit either of those countries. This accounts for 85 of the 100, so the remaining 15 of them must have visited Germany but not Spain. (That is, the numbers on the 4 third level branches which follow the $F^C$ branch must sum to 100, so the missing number must be 15.)

Now, we can see that, among the tourists who did not visit France, $40+15 = 55$ did visit Germany. Adding these to the 80 that visited both France and Germany, we see that 135 of the tourists visited Germany.

**Question 4**

To find $n(F^C \cup S)$, we can add up the numbers on all third level $S$ branches which are in paths containing either an $F^C$ branch or an $S$ branch (or both). In order to do this, we need to fill in the rest of the numbers on third level branches. Since there were 80 who went to France and Germany, of whom 10 did not go to Spain, then 70 of them did visit Spain. Likewise, of the 200 who went to France, 80 went to Germany, so 120 did not. Of these, 30 also did not go to Spain, so the other 90 did visit Spain. We have:




There are 6 paths through the tree that include an $F^C$ branch, an $S$ branch or both. (All of the paths which end with an $S$ branch, and all of the paths which start with the $F^C$ branch.) Adding up the numbers on the terminal branches of these paths we get:

$$Pr[F^C \cup S] = 70 + 90 + 40 + 15 + 45 + 0 = 260$$

Another Approach

Of course, we can do this simply using the formula $n(F^C \cup S) = n(F^C) + n(S) - n(F^C \cap S)$. For that, we already know that $n(F^C) = 100$ (calculated in question 3), but we need to find $n(S)$, for which we need to fill in the missing numbers on $S$ branches, as described above. We get

$$n(S) = 70 + 90 + 40 + 45 = 245$$

Now, to get $n(F^C \cap S)$, we add up the numbers on the $S$ branches of the 2 paths which include both an $F^C$ branch and an $S$ branch:

$$n(F^C \cap S) = 40 + 45 = 85$$

Thus we get

$$n(F^C \cup S) = n(F^C) + n(S) - n(F^C \cap S) = 100 + 245 - 85 = 260$$

Quicker Approach

It is easier to realize that the only tourists who are not in the set $F^C \cup S$ are those who did visit France and did not visit Spain, i.e. those who are in $F \cap S^C$. We can easily find $n(F \cap S^C)$ as

$$n(F \cap S^C) = n(F \cap G \cap S^C) + n(F \cap G^C \cap S^C) = 10 + 30 = 40$$

That is, doing this we only need to use 2 numbers which were given to us on the tree, along with $n(U)$, which was also given to us. We get

$$n(F^C \cup S) = n([F \cap S^C]^C) = n(U) - n(F \cap S^C) = 300 - 40 = 260$$

**Question 5**

We wish to construct 5-digit numbers, using only digits from the set $\{2, 3, 4, 5, 6, 7, 8\}$, where repetition is allowed. The only restriction is that we want all of the numbers to be greater than 35000.

Clearly, a 5-digit number which starts with a 2 is not greater than 35000, so 2 cannot be used as the first digit. A 5-digit number starting with a 4, 5, 6, 7 or 8 must be greater than 35000, so all of these may be used as the first digit. For a 5-digit number starting with 3, some are greater than 35000 but some are not. Therefore we must treat this as a separate case.

**Case 1:** first digit is 3
There is only 1 way to choose the first digit, since it must be 3. Also, in order to be forming only numbers which are greater than 35000, the second digit must be at least 5, i.e. must be 5, 6, 7 or 8, so there are 4 choices available for the second digit. Now, any of the 7 given digits can be used for the third digit, and likewise for the fourth and fifth digits. Therefore there are

$$1 \times 4 \times 7 \times 7 \times 7 = 4 \times 7^3$$

different 5-digit numbers greater than 35000 that can be formed using the given digits, when the first digit is 3.

**Case 2** first digit is bigger than 3
There are 5 choices available for the first digit: 4, 5, 6, 7 or 8. Since this ensures that the number is bigger than 35000, then any of the 7 given digits may be used for any of the second, third, fourth and fifth digits of the number being constructed. This gives

$$5 \times 7 \times 7 \times 7 \times 7 = 5 \times 7^4$$

more 5-digit numbers greater than 35000 that can be formed with the given digits.

In total, there are $4 \times 7^3 + 5 \times 7^4$ such numbers.

**Question 6**

We need to count the number of ways in which 3 girls and 6 boys can be arranged in a line with the girls all together. There are 3! ways to arrange the girls and 6! ways to arrange the boys, and then there are 7 places in the line of boys into which the group of girls could be put (i.e. before first, second, ..., or sixth boy, or after the sixth boy). Therefore there are $3! \times 6! \times 7 = 3! \times 7!$ different arrangements possible.

Another Approach

The girls can be arranged within their group in $3!$ different ways. Now, we arrange the six boys plus one group of 3 girls into a line, which can be done in $7!$ different ways. (That is, we consider the group of girls as a single object and arrange 7 objects in a line.) This gives a total of $3! \times 7!$ different arrangements possible.

**Question 7**

If there were no restriction, then each clerk could be assigned to either of the 2 stores, without any consideration of where the other clerks are sent. This means that each of the 10 clerks would correspond to a decision of the form 'where should this clerk be sent?', with 2 choices available for each of these 10 decisions. Therefore, if there were no restrictions, there would be $2^{10}$ different ways to assign the clerks.

We do, however, have a restriction here. We are told that each store must receive at least one clerk. How many of the assignments we calculated do not satisfy this requirement? Only the assignment in which all 10 clerks are sent to the first store, and the one in which all 10 clerks are sent to the second store. Therefore 2 of the assignments we counted should not have been allowed, so we see that there are $2^{10} - 2$ assignments which do satisfy the requirement.

*Notice:* If we attempt to do this problem by direct counting, rather than indirect counting, it becomes very complicated. If we assign one clerk to each store, to ensure that the restriction is met, and then consider how the remaining 8 clerks are assigned, then we are double counting, because we have partially ordered the clerks who are sent to a particular store, by considering the 'first' clerk assigned to a store to be in some way different from the others. We would need to compensate for this double counting (and that's the part that's quite complicated).

**Question 8**

We need to count the number of circular permutations of 8 people. This is given by $(8-1)! = 7!$.

**Question 9**

We want to count the number of ways in which 8 people can be arranged so that 5 of them are in a circle and the other 3 are in a line. There are $\binom{8}{5}$ ways to decide which 5 will sit at the round table. Next, there are $(5-1)! = 4!$ ways for these 5 to arrange themselves at the table. Finally, the other 3 can arrange themselves in 3! different ways to sit along the wall. Therefore the total number of possible arrangements is

$$\binom{8}{5} \times 4! \times 3!.$$

**Question 10**

As in question 7, we want to avoid the complications of partial ordering that arise if we try to count directly. (And the tedium that will result if we use various cases for the various possible numbers of men on the committee.) Therefore we approach this by counting the total number of possible committees and also counting the number which do not contain at least one man.

Since there are 16 people in total, and the committee is to consist of 5 of them, then there are $\binom{16}{5}$ different ways to form the committee without regard for how many men or women there may be on it. Also, since there are 9 women and 7 men, there are $\binom{9}{5}$ different committees which could be formed from only the women, i.e. without any men. Therefore there are $\binom{16}{5} - \binom{9}{5}$ different ways to form a committee with at least one man.

**Question 11**

We can think of this problem as choosing 10 of the 20 men who will be paired with women, and then assigning each of these men to one of the women, i.e. arranging the men. (Alternatively, we could think of it as arranging the women. But we only need to arrange one group or the other in order to pair them up.) Therefore the number of different ways in which this can be done is

$$\binom{20}{10} 10!$$

**Question 12**

Since the 3 identical-sized groups will be doing the same thing, these are indistinguishable groups. Therefore when we use the multinomial coefficient to divide the 12 students into 3 groups of 4, we must 'undo' the ordering of the groups by dividing by the 3! possible arrangements of the 3 groups. Therefore the number of ways in which the students can be divided up is

$$\binom{12}{4\ 4\ 4} \div 3! = \frac{1}{3!}\binom{12}{4\ 4\ 4}$$

**Question 13**

Since Charles and Henry must not belong to the same group, then they must be in different groups. That is, the 3 groups will consist of one containing Charles, one containing Henry and one that contains neither. Therefore the other 10 students must be divided up into 3 to join Charles' group, 3 to join Henry's group and 4 to form the other group. (Note that Charles' group and Henry's group are distinguishable.) The number of ways of forming the groups is now

$$\binom{10}{3\ 3\ 4}$$

**Question 14**

We are simply looking for the number of permutations of 10 specified digits, where there is repetition among the specified digits. This is the same sort of thing as permuting the letters of a word, i.e., this is a labelling problem. We want to count the number of ways to assign labels to the 10 positions of the 10 digit number, using 3 labels that say '1', 3 labels that say '2', 2 labels that say '3' and one label each saying '4' and '5'. The number of ways in which this can be done is

$$\binom{10}{3\ 3\ 2\ 1\ 1}$$

**Question 15**

Let $S$ be the set of all ways of choosing 2 cards from a standard deck. Then we have

$$n(S) = \binom{52}{2} = \frac{52!}{2!50!} = \frac{52 \times 51}{2} = 26 \times 51$$

Also, let $E$ be the event that neither of the chosen cards is a Heart. Since there are 39 cards in the deck which are not Hearts, then the number of ways of choosing 2 non-Hearts is

$$n(E) = \binom{39}{2} = \frac{39!}{2!37!} = \frac{39 \times 38}{2} = 39 \times 19$$

We see that the probability 2 non-Hearts are drawn is

$$Pr[E] = \frac{n(E)}{n(S)} = \frac{39 \times 19}{26 \times 51} = \frac{3 \times 13 \times 19}{2 \times 13 \times 3 \times 17} = \frac{19}{34}$$

**Question 16**

We have 6 customers. Each of these must choose one of the 2 salespersons. Thus there are 6 decisions to be made, each involving 2 choices. Letting $S$ be the number of ways in which the 6 customers can choose salespeople, we have

$$n(S) = 2 \times 2 \times ... \times 2 = 2^6$$

If exactly 3 of the customers choose salesperson $A$, then the other 3 must choose salesperson $B$. Letting $E$ be the event that exactly 3 of the 6 choose $A$, we see that the number of different ways this can occur is simply the number of different ways of choosing 3 of the 6 people to be the ones who choose $A$. Therefore we see that

$$n(E) = \binom{6}{3} \qquad \text{so that} \qquad Pr(E) = \frac{n(E)}{n(S)} = \frac{\binom{6}{3}}{2^6}$$

**Question 17**

Let $S$ be the set of all permutations of the 7 letters. Then we have $n(S) = 7!$.

Let $R$ be the event that the letters $A$, $B$ and $C$ appear all together. We can consider these letters to be a single unit. Then we have 5 objects to arrange in a line: $D$, $E$, $F$, $G$, and the unit which consists of $A$, $B$ and $C$. There are 5! arrangements of these 5 objects possible, as well as 3! ways in which the 3 objects $A$, $B$ and $C$ can be arranged within their group, so we have $n(R) = 5!3!$ and we get $Pr[R] = \frac{5!3!}{7!}$.

Another Approach:

We can think of arranging the letters $D$, $E$, $F$ and $G$ in a line, and then putting the group containing $A$, $B$ and $C$ into the line. There are 4! ways to arrange the $D$, $E$, $F$ and $G$, then 5 choices of where to put the group of 3 other letters (i.e. before any of the 4 letters, or after the last letter), as well as 3! ways to arrange those other 3 letters within their group. Then we have

$$n(R) = 4! \times 5 \times 3! = (5 \times 4!) \times 3! = 5!3!$$

so once again we get $Pr[R] = \frac{5!3!}{7!}$.

**Question 18**

The easiest way to approach this is to think about the complementary event – that Haley does not select any of the first 3 questions. In that case, Haley must select 5 of the last 7 questions, which can be done in $\binom{7}{5}$ ways. Letting $S$ be the set of all ways that Haley could choose 5 of the 10 questions to answer, and $E$ be the event that Haley chooses at least one of the first 3 questions, we have $n(S) = \binom{10}{5}$ and $n(E^C) = \binom{7}{5}$, so we get

$$Pr[E] = 1 - Pr[E^C] = 1 - \frac{n(E^C)}{n(S)} = 1 - \frac{\binom{7}{5}}{\binom{10}{5}}$$

**Question 19**

We know that $Pr[E \cap F]$ can be found using the formula

$$Pr[E \cap F] = Pr[E|F] \times Pr[F] = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$$

**Question 20**

Easiest Approach:

We know that the same number came up on all 3 dice. There are 6 numbers that it might have been, and all are equally likely. Therefore the probability that the number which came up on all 3 dice was a 6, giving a sum of 18, is 1/6.

The Long Way:

Let $S$ be the set of all possible outcomes of tossing 3 dice. Then each element of $S$ is an ordered triple of the numbers that came up on the 3 dice (i.e. the first die, the second die and the third die). Since there are 6 possibilities for each, we have $n(S) = 6 \times 6 \times 6$.

Let $A$ be the event that the 3 dice are all the same, and let $E$ be the event that the sum of the 3 dice is 18. Then we are looking for $Pr[E|A] = \frac{Pr[E \cap A]}{Pr[A]}$. In $S$, there are 6 outcomes which have the same number 3 times (one for each of the 6 possible numbers which might come up all 3 times), so $n(A) = 6$ and $Pr[A] = \frac{6}{6^3}$. Also, in the set $S$ there is only 1 outcome which has all 3 numbers the same and also has the sum of the numbers being 18, i.e. the outcome $(6, 6, 6)$. Therefore $n(E \cap A) = 1$ so that $Pr[E \cap A] = \frac{1}{6^3}$. Thus we get

$$Pr[E|A] = \frac{Pr[E \cap A]}{Pr[A]} = \frac{1/6^3}{6/6^3} = \frac{1}{6^3} \times \frac{6^3}{6} = \frac{1}{6}$$

**Question 21**

In order to determine whether or not $E$ and $F$ are independent events, we check whether or not $Pr[E \cap F]$ is equal to $Pr[E] \times Pr[F]$. We have

$$Pr[E] \times Pr[F] = \frac{5}{8} \times \frac{2}{5} = \frac{2}{8} = \frac{1}{4} = Pr[E \cap F]$$

so we see that $E$ and $F$ **are** independent events.

**Question 22**

To find $Pr[E]$, we sum the path probabilities for all paths in which event $E$ occurs. We get

$$Pr[E] = (.4)(.5) + (.6)(1) = .20 + .6 = .8$$

Another way to look at it

From the tree structure, we see that events $A_1$ and $A_2$ partition the sample space. Thus we can partition event $E$ according to these two events:

$$Pr[E] = Pr[E \cap A_1] + Pr[E \cap A_2]$$

Now, we use the fact that for any events $C$ and $D$, $C \cap D$ can be calculated as $Pr[C|D] \times Pr[D]$. This gives

$$Pr[E] = Pr[E|A_1] \times Pr[A_1] + Pr[E|A_2] \times Pr[A_2]$$

Now, we simply need to recognize that these probabilities are all given on the probability tree. We get:

$$Pr[E] = Pr[E|A_1] \times Pr[A_1] + Pr[E|A_2] \times Pr[A_2] = (.5)(.4) + (1)(.6) = .2 + .6 = .8$$

Notice that this was not in fact any different from the first approach. When we calculate a path probability, we are simply using the formula $Pr[C \cap D] = Pr[C|D] \times Pr[D]$, and when we find the probability of a particular event by adding up various path probabilities, we are in fact using partitioning. (That is, the tree has partitioned the event.)

**Question 23**

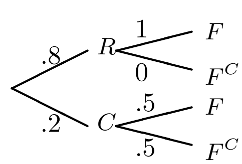
$Pr[E|A_1]$ is simply the probability shown on the tree, on the $E$ branch which follows the $A_1$ branch. Therefore $Pr[E|A_1] = .5$.

**Question 24**

$Pr[E \cap A_1]$ is the path probability for the path which contains an $A_1$ branch followed by an $E$ branch. As we saw before, we have $Pr[E \cap A_1] = (.4)(.5) = .2$.

**Question 25**

We use a probability tree to model what's happening in the stochastic process, to help sort it all out. We can define the following events: $R$ - the professor does research; $C$ - the professor cleans his office; and $F$ - the professor forgets his keys (so that $F^C$ is the event that he does not forget his keys). We are told that $Pr[R] = .8$, $Pr[C] = .2$. Also, event $F$ always happens when $R$ happens, so $Pr[F|R] = 1$, while event $F$ only happens half the time when $C$ happens, so that $Pr[F|C] = .5$. We get the following probability tree:




We want to determine the probability that the professor did research last Tuesday, given the information that he forgot his keys that day. Thus, we are looking for $Pr[R|F]$. This is a Bayes' Theorem type of question, since we are looking for the probability of something which comes earlier in the tree, given information about something which happens later in the tree. We get:

$$\begin{aligned} Pr[R|F] &= \frac{\text{path prob. of the path containing both } R \text{ and } F}{\text{sum of path prob.'s of all paths in which } F \text{ occurs}} \\ &= \frac{(.8)(1)}{(.8)(1)+(.2)(.5)} \\ &= \frac{.8}{.8+.1} \\ &= \frac{8}{9} \end{aligned}$$

**Question 26**

This time, we want to calculate the probability that the professor will remember (i.e. not forget) his keys on a particular day, without any information about whether the professor does research or cleans his office that day, so we simply need to find $Pr[F^C]$. We get this by summing the path probabilities for all paths through the tree which include an $F^C$ branch. We have:

$$Pr[F^C] = (8.)(0) + (.2)(.5) = 0 + .1 = .1 = \frac{1}{10}$$

**Question 27**

When a single die is tossed, the probability that a 5 or a 6 shows is $\frac{2}{6} = \frac{1}{3}$. When 3 tosses are made, we are performing Bernoulli trials, since each toss of a die is independent of what happened on any earlier toss. Thus if we define success to be that a 5 or a 6 is tossed, we have a series of $n = 3$ Bernoulli trials in which the probability of success is $p = \frac{1}{3}$. $E$ is the event that there is at least one success in the 3 trials. Of course, the easiest way to calculate the probability of at least one success is by considering the complementary event, that no successes are observed. We see that

$$Pr[E] = 1 - Pr[E^C] = 1 - Pr[0 \text{ successes}] = 1 - \binom{3}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 = 1 - \frac{8}{27} = \frac{19}{27}$$

**Question 28**

Each of the computers that George tries to install could end up in any of 3 ways: the installation could be completely successful ($S$), or could be partially successful ($P$), or the computer could be damaged beyond repair ($D$). We are told that $Pr[S] = .5$, $Pr[P] = .3$ and $Pr[D] = .2$. George attempting to install 10 computers corresponds to $n = 10$ independent trials of the experiment 'George tries to install a computer'. The probability that outcome $D$ occurs 7 times and outcome $S$ occurs only once, so that outcome $P$ must have occurred on the other $10 - (7 + 1) = 2$ trials, is given by

$$\binom{10}{7\ 2\ 1}(.2)^7(.3)^2(.5)^1$$

**Question 29**

The probability that event $D$ occurs on all of the 10 trials is

$$\binom{10}{0\ 0\ 10}(.5)^0(.3)^0(.2)^{10} = 1 \times 1 \times 1 \times (.2)^{10} = (.2)^{10}$$

**Question 30**

Having at least one computer being damaged beyond repair is the complement of having no computers damaged beyond repair. That is, if we consider 'outcome $D$ is observed' to be a success, then both outcomes $S$ and $P$ are failures, so we have $p = .2$ and $q = .8$. In a series of $n = 10$ independent (Bernoulli) trials, the probability that no successes are observed is given by

$$\binom{10}{0}(.2)^0(.8)^{10} = (.8)^{10}$$

ans so the probability of the complementary event, i.e. that at least one computer is damaged beyond repair, is $1 - (.8)^{10}$.

**Question 31**

The event $(X = 1)$ contains all sample points in which there is exactly one club in the hand, and hence also 2 non-clubs. There are $\binom{13}{1}$ ways to choose one of the 13 clubs, and $\binom{39}{2}$ ways to choose 2 of the 39 non-clubs, so this event contains $\binom{13}{1} \times \binom{39}{2}$ different sample points.

**Question 32**

$$Pr[X > 0] = Pr[X = 1] + Pr[X = 2] = \frac{4}{9} + \frac{1}{3} = \frac{7}{9}$$

Another Approach:
Since the event $(X > 0)$ includes all of the possible values of $X$ except for the value 0, we have

$$Pr[X > 0] = 1 - Pr[X = 0] = 1 - \frac{2}{9} = \frac{7}{9}$$

**Question 33**

We use the formula for expected value of a discrete random variable:

$$E(X) = 0 \times Pr[X = 0] + 1 \times Pr[X = 1] + 2 \times Pr[X = 2] = 0 \times \frac{2}{9} + 1 \times \frac{4}{9} + 2 \times \frac{1}{3} = 0 + \frac{4}{9} + \frac{2 \times 3}{9} = \frac{10}{9}$$

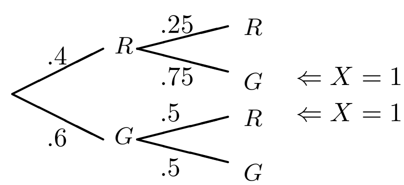
**Question 34**

We use the fact that for any possible value of the random variable $X$, the probability that $X$ has exactly that value is given by the probability that $X$ is no more than that value, less the probability that $X$ is no more than the next smaller possible value. Here, the next smaller possible value, from 5, is 4, so we get:

$$Pr[X = 5] = Pr[X \leq 5] - Pr[X < 5] = Pr[X \leq 5] - Pr[X \leq 4] = \frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12} = \frac{5}{12}$$

**Question 35**

We start by drawing a probability tree to model the stochastic process. The possible outcomes on each draw are $R$ - a red disk is drawn, and $G$ - a green disk is drawn. At the time of the first draw, 2 of the 5 disks in the bag are red, while the other 3 are green, so a red disk will be drawn with probability $\frac{2}{5} = .4$ and a green disk will be drawn with probability $\frac{3}{5} = .6$. Since the 2 draws are performed without replacement, there are only 4 disks in the bag at the time of the second draw. The proportions of red and green disks in the bag depend on the colour of the disk which was already removed from the bag on draw 1. We get:




$X$ is defined to be the number of red disks drawn, which may be 0, 1 or 2. The event $(X = 1)$ occurs on all paths through the tree in which there is one $R$ branch and one $G$ branch, as shown above. We get

$$Pr[X = 1] = (.4)(.75) + (.6)(.5) = .3 + .3 = .6$$

**Question 36**

Since $\frac{1}{5} + \frac{4}{5} = 1$, we see that these are the only 2 possible values of $X$. We first need to calculate the mean:

$$\mu = \left(0 \times \frac{1}{5}\right) + \left(1 \times \frac{4}{5}\right) = 0 + \frac{4}{5} = \frac{4}{5}$$

Now, we need to find the variance. We can use either of 2 formulas for this. Using the easier formula, we have

$$V(X) = E(X^2) - \mu^2 = \left[\left(0^2 \times \frac{1}{5}\right) + \left(1^2 + \frac{4}{5}\right)\right] - \left(\frac{4}{5}\right)^2 = \frac{4}{5} - \frac{16}{25} = \frac{20}{25} - \frac{16}{25} = \frac{4}{25}$$

Alternatively, using the formula from the definition of $V(X)$ we get:

$$\begin{aligned} V(X) &= E[(X-\mu)^2] = \left(0-\frac{4}{5}\right)^2 \times \frac{1}{5} + \left(1-\frac{4}{5}\right)^2 \times \frac{4}{5} \\ &= \left(-\frac{4}{5}\right)^2 \times \frac{1}{5} + \left(\frac{1}{5}\right)^2 \times \frac{4}{5} \\ &= \frac{16}{25} \times \frac{1}{5} + \frac{1}{25} \times \frac{4}{5} \\ &= \frac{16+4}{25 \times 5} = \frac{20}{5 \times 25} = \frac{4}{25} \end{aligned}$$

Of course, we were asked to find the standard deviation, not the variance. We get

$$\sigma(X) = \sqrt{V(X)} = \sqrt{\frac{4}{25}} = \frac{2}{5}$$

**Question 37**

We have a discrete random variable $X$ with the probability distribution function

| $x$ | $Pr[X = x]$ |
|---|---|
| $-2$ | $\frac{1}{2}$ |
| $1$ | $\frac{1}{2}$ |

(Note: We can tell that these are the only possible values of $X$ because their probabilities sum to 1.)

We are asked to find $V(X) = E(X^2) - \mu^2$, so we need to find $\mu$ and $E(X^2)$.

$$\begin{aligned} \mu &= (-2)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) = -1 + \frac{1}{2} = -\frac{1}{2} \\ E(X^2) &= (-2)^2\left(\frac{1}{2}\right) + (1)^2\left(\frac{1}{2}\right) = (4)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) = 2 + \frac{1}{2} = \frac{5}{2} \\ \text{So } V(X) &= \left(\frac{5}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{5}{2} - \frac{1}{4} = \frac{10}{4} - \frac{1}{4} = \frac{9}{4} \end{aligned}$$

**Question 38**

In this problem, each possible value of the discrete r.v. $X$ occurs with only one value of the discrete r.v. $Y$. For instance, $(X = 0)$ happens whenever outcome $t_1$ occurs, and when this happens, the value of $Y$ is 5, i.e. $(Y = 5)$ occurs. So whenever $t_1$ occurs, the event $[(X = 0) \cap (Y = 5)]$ occurs, so $Pr[(X = 0) \cap (Y = 5)] = Pr[t_1] = .5$.

We are asked to find the value of $Pr[(X = 1) \cap (Y = 5)]$, i.e. the probability that $X$ has the value 1 at the same time that $Y$ has the value 5. This is not a combination that happens with any of the possible outcomes of the experiment. That is, $(X = 1)$ only happens when outcome $t_2$ occurs, and $(Y = 5)$ only happens when outcome $t_1$ occurs. Since these never happen at the same time, we *cannot* have $X = 1$ at the same time as $Y = 5$, so $Pr[(X = 1) \cap (Y = 5)] = 0$.

**Question 39**

When outcome $t_1$ occurs, we have $X = 0$ and $Y = 5$, so $XY = 0 \times 5 = 0$. When $t_2$ occurs, we have $X = 1$ and $Y = 10$, so $XY = 10$. When $t_3$ occurs, we have $X = 2$ and $Y = 20$, so $XY = 40$. Therefore $XY = 0$ only when outcome $t_1$ occurs, and the probability of this happening is .5, so $Pr[XY = 0] = .5$.

**Question 40**

We use the $XY$ values found above, and the probabilities of the various outcomes $t_1$, $t_2$ and $t_3$ which provide those values of $XY$. That is, when $t_1$ occurs, $XY = 0$, and this happens with probability .5; when $t_2$ occurs, $XY = 10$, and this happens with probability .3; and when $t_3$ occurs, $XY = 40$, and the probability that this happens is .2, so we get:

$$E(XY) = (0)(.5) + (10)(.3) + (40)(.2) = 0 + 3 + 8 = 11$$

**Question 41**

We use the fact that $E(aX + bY) = aE(X) + bE(Y)$, with $a = 5$ and $b = -3$.

$$E(5X - 3Y) = 5 \times E(X) + (-3) \times E(Y) = 5 \times 3 - 3 \times (-2) = 15 - (-6) = 21$$

**Question 42**

We know that $V(aX + b) = a^2V(X)$ and so $\sigma(aX + b) = |a|\sigma(X)$. Here, we have $a = 2$ and $b = -1$. We get

$$\sigma(Y) = \sigma(2X - 1) = |2|\sigma(X) = 2 \times \sqrt{9} = 2 \times 3 = 6$$

**Question 43**

Since $X$ is the number of successes observed in $n$ Bernoulli trials with probability of success $p$, then $X = B(n, p)$. That is, this defines $X$ to be a binomial random variable, so $X$ has a binomial distribution. Since $X$ counts the number of successes observed, which cannot be negative, and must be integer, then the value of $X$ must be a non-negative integer. (That is, the possible values of $X$ are 0, 1, 2, ... and $n$, which are all non-negative integers.) Of course, we know that the mean of $B(n, p)$ is $\mu = np$, and that the variance of $B(n, p)$ is $np(1-p)$.

Since $X$ is a binomial random variable, then $X$ is a discrete random variable. That is, by defining that $X$ is the number of successes in a series of Bernoulli trials, we have defined that the only possible values of $X$ are (as previously observed) 0, 1, ... and $n$, which means that there are only finitely many possible values which $X$ can have and hence $X$ is a discrete random variable. A *normal* random variable, on the other hand, is a continuous random variable, which may take on any real value. And $X$ is not a continuous random variable, so it cannot be a normal random variable. We know that, if certain conditions on $n$ and $p$ are satisfied, then the binomial random variable $X$ will be *approximately normal*, but that's not the same as *being* a normal random variable.

## Question 44

First, we observe that since the given probabilities sum to 1, these must be the only possibilities. Therefore the only possible values of $X$ are 0 and 1, and the only possible values of $Y$ are 1 and 2. We can find the probability that $X = 0$ by considering $X$ to be partitioned according to $Y$-value, i.e. we have

$$Pr[X = 0] = Pr[(X = 0) \cap (Y = 1)] + Pr[(X = 0) \cap (Y = 2)] = .2 + .1 = .3$$

Similarly, we consider a partition of $(Y = 1)$:

$$Pr[Y = 1] = Pr[(X = 0) \cap (Y = 1)] + Pr[(X = 1) \cap (Y = 1)] = .2 + .4 = .6$$

Notice that we have

$$Pr[X = 0] \times Pr[Y = 1] = (.3)(.6) = .18 \neq Pr[(X = 0) \cap (Y = 1)]$$

This tells us that $X$ and $Y$ are not independent random variables, they are dependent random variables.

*Notice:* We can organize the calculations shown above by constructing the joint distribution table for $X$ and $Y$. We have:

| | $(Y = 1)$ | $(Y = 2)$ | $Pr[X = x]$ |
|---|---|---|---|
| $(X = 0)$ | .2 | .1 | .3 |
| $(X = 1)$ | .4 | .3 | .7 |
| Pr[Y = y] | .6 | .4 | |

## Question 45

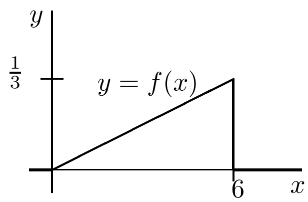
We know that if $X$ and $Y$ are independent random variables, then $V(X + Y) = V(X) + V(Y)$. Here, we get

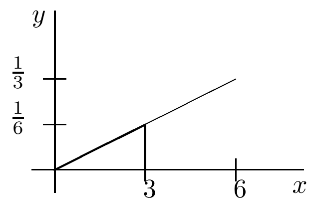
$$V(X + Y) = V(X) + V(Y) = \frac{1}{4} + \frac{1}{25} = .25 + .04 = .29$$

Therefore, we have $\sigma(X + Y) = \sqrt{V(X + Y)} = \sqrt{.29}$.

**Question 46**

We first draw the density function, which runs along the $x$-axis except between $x = 0$ and $x = 6$. On this interval, it is the line segment joining the points $(0, f(0)) = (0, \frac{0}{18}) = (0, 0)$ and $(6, f(6)) = (6, \frac{6}{18}) = (6, \frac{1}{3})$. The value of $Pr[X < 3]$ will be given by the area under this curve on the interval $x < 3$, i.e. the area of the region lying below the $y = f(x)$ and above the $x$-axis from $x = 0$ to $x = 3$.




We see that the region is a triangle. The base (width) of the triangle is the distance from 0 to 3, i.e. 3, and the height of the triangle is the height of the function at $x = 3$, i.e. $f(3) = \frac{3}{18} = \frac{1}{6}$. Therefore we have

$$Pr[X < 3] = \frac{1}{2} \times 3 \times \frac{1}{6} = \frac{1}{4}$$

**Question 47**

We see that the possible values of $X$ are evenly spaced, $k = 2$ units apart. Therefore we must apply a continuity correction of $\frac{k}{2} = 1$. So the event $(X \leq 4)$ is approximated by the event $(Y < 4 + \frac{k}{2}) = (Y < 4 + 1) = (Y < 5)$ and we see that $Pr[X \leq 4]$ is approximated by $Pr[Y < 5]$.

**Question 48**

This time, since the possible values of $X$ are evenly spaced $k = 1$ unit apart, the continuity correction is $\frac{k}{2} = .5$. We know that the event $(a \leq X \leq b)$ would be approximated by an event in terms of $Y$ obtained by extending the interval from $a$ to $b$ by $\frac{k}{2}$ in each direction. That is, $(a \leq X \leq b)$ is approximated by $(a - .5 < Y < b + .5)$. In this case, we have

$$(.5 < Y < 1.5) = (a - .5 < Y < b + .5)$$

so we get $a - .5 = .5 \Rightarrow a = 1$ and $b + .5 = 1.5 \Rightarrow b = 1$. That is, we see that the event $(.5 < Y < 1.5)$ is the approximation for the event $(1 \leq X \leq 1)$, which is just the event $(X = 1)$. Therefore $Pr[.5 < Y < 1.5]$ gives the value of $Pr[X = 1]$.

**Question 49**

We need the form $Pr[Z < k]$ in order to use the $Z$-table. We use the complementary event. That is, we know that $Pr[Z > 1.54] = 1 - Pr[Z < 1.54]$. From the $Z$-table, we see that $Pr[Z < 1.54] = .9382$, so we get $Pr[Z > 1.54] = 1 - .9382 = .0618$.

**Question 50**

One Approach:
We use the fact that $Pr[a < Z < b] = Pr[Z < b] - Pr[Z < a]$. We will also use the symmetry of the $Z$-curve to recognize that $Pr[Z < -k] = Pr[Z > k]$, as well as needing complementation again. We get

$$\begin{aligned} Pr[-1.2 < Z < 1.2] &= Pr[Z < 1.2] - Pr[Z < -1.2] \\ &= Pr[Z < 1.2] - Pr[Z > 1.2] \\ &= Pr[Z < 1.2] - (1 - Pr[Z < 1.2]) \\ &= Pr[Z < 1.2] - 1 + Pr[Z < 1.2] \\ &= 2 \times Pr[Z < 1.2] - 1 \\ &= 2(.8849) - 1 = 1.7698 - 1 = .7698 \end{aligned}$$

A Quicker Approach:
We have seen before that $Pr[-k < Z < k] = 2Pr[Z < k] - 1$. We can simply remember and use this formula, with $k = 1.2$ this time, to skip most of the steps above:

$$Pr[-1.2 < Z < 1.2] = 2Pr[Z < 1.2] - 1 = 2(.8849) - 1 = .7698$$

**Question 51**

We first want to re-express the given piece of information. We use the fact that $Pr[k < Z < 1.3] = Pr[Z < 1.3] - Pr[Z < k]$, to see that

$$\begin{aligned} Pr[k < Z < 1.3] = .2853 &\Rightarrow Pr[Z < 1.3] - Pr[Z < k] = .2853 \\ &\Rightarrow Pr[Z < k] = Pr[Z < 1.3] - .2853 \end{aligned}$$

From the $Z$-table, we see that $Pr[Z < 1.3] = .9032$, so we have

$$Pr[Z < k] = .9032 - .2853 = .6179$$

Looking to the $Z$-table once more, we see that $.6179 = Pr[Z < 0.30]$, so $Pr[Z < k] = Pr[Z < 0.30]$ and we see that we must have $k = 0.30$.

**Question 52**

We are told that $X$ is normally distributed, with mean $\mu = 100$ and standard deviation $\sigma = 5$. We must find $Pr[X < 106]$. We get:

$$\begin{aligned} Pr[X < 106] &= Pr\left[Z < \frac{106-\mu}{\sigma}\right] \\ &= Pr\left[Z < \frac{106-100}{5}\right] \\ &= Pr\left[Z < \frac{6}{5}\right] \\ &= Pr[Z < 1.2] \\ &= .8849 \end{aligned}$$

**Question 53**

We use the fact that $Z = \frac{X-\mu}{\sigma}$, so that $Pr[X < k] = Pr\left[Z < \frac{k-\mu}{\sigma}\right]$. We get

$$\begin{aligned} Pr[990 < X < 1020] &= Pr[X < 1020] - Pr[X < 990] \\ &= Pr\left[Z < \frac{1020-1000}{10}\right] - Pr\left[\frac{990-1000}{10}\right] \\ &= Pr[Z < 2] - Pr[Z < -1] \\ &= Pr[Z < 2] - Pr[Z > 1] \\ &= Pr[Z < 2] - (1 - Pr[Z < 1]) \\ &= Pr[Z < 2] + Pr[Z < 1] - 1 \\ &= .9772 + .8413 - 1 = .8185 \end{aligned}$$

**Question 54**

The number of candies in the carton must be integer valued, so if we let $X$ be the number of Wriggly Jigglies in a Super Jumbo carton, $X$ is a discrete random variable. The possible values of $X$, which are consecutive integers, are evenly spaced $k = 1$ unit apart. We are told that $X$ is approximately normal, which means that the Normal random variable with the same mean and standard deviation as $X$ is a good approximation for $X$. As always, when we use a normal r.v. (which is continuous) to approximate a discrete r.v., we need to apply a continuity correction, in this case $\frac{k}{2} = .5$. Letting $Y$ be the normal r.v. with $\mu = 60$ and $\sigma = 5$, we use $Y$ to approximate $X$.
We want to know the probability that a Super Jumbo carton contains more than 65 Wriggly Jigglies, i.e. $Pr[X > 65]$. Of course, containing more than 65 candies is the complement of containing no more than 65 candies. That is, $Pr[X > 65] = 1 - Pr[X \leq 65]$. We get:

$$\begin{aligned} Pr[X > 65] &= 1 - Pr[X \leq 65] = 1 - Pr[Y < 65.5] \\ &= 1 - Pr\left[Z < \frac{65.5 - 60}{5}\right] = 1 - Pr[Z < 1.1] \\ &= 1 - .8643 = .1357 \end{aligned}$$

**Question 55**

A fair coin is being tossed repeatedly and we are interested in whether or not Heads comes up each time. Since repeated tosses of a coin are independent, we are performing $n = 100$ Bernoulli trials in which success is defined as Heads being tossed, so $p = .5$ .

$X$ is defined to be the number of times that Heads comes up, so $X$ is a Binomial r.v., i.e. is counting the number of successes in $n = 100$ Bernoulli trials with $p = .5$, so $X = B(100, .5)$. We want to find $Pr[X = 50]$, which is the probability of observing exactly $k = 50$ successes on $n = 100$ trials with $p = .5$. This, of course, is given by:

$$\binom{100}{50}(.5)^{50}(1 - .5)^{50} = \binom{100}{50}(.5)^{100} = .079589237...$$

If your calculator won't do this calculation (mine will do 100 choose 50, but won't do 100!, if I try to do the 100 choose 50 calculation myself), we can use a normal approximation to the binomial distribution. Since $X = B(100, .5)$, and we have $np = 50$ and $nq = 50$ which are both bigger than 5, we know that the Normal r.v. with the same mean and standard deviation as $X$ is a good approximation for $X$. We have $\mu = np = 50$ and $\sigma = \sqrt{npq} = \sqrt{50(.5)} = \sqrt{25} = 5$. Let $Y$ be the normal random variable with $\mu = 50$ and $\sigma = 5$. Using a continuity correction of $\frac{k}{2} = \frac{1}{2}$, as always, we have:

$$\begin{aligned} Pr[X = 50] &= Pr\left[50 - \frac{1}{2} < Y < 50 + \frac{1}{2}\right] \\ &= Pr[49.5 < Y < 50.5] \\ &= Pr\left[\frac{49.5 - 50}{5} < Z < \frac{50.5 - 50}{5}\right] \\ &= Pr\left[-\frac{.5}{5} < Z < \frac{.5}{5}\right] \\ &= Pr[-.1 < Z < .1] \\ &= 2 \times Pr[Z < .1] - 1 \\ &= 2(.5398) - 1 \\ &= .0796 \end{aligned}$$