Student's Name [print] Student Number

Mathematics 028b Final Examination **CODE 111** **Friday, April 23, 2004**

7:00 p.m. - 10:00 p.m.

1. In a poll, it was discovered that all of the students in a certain High School class have a VCR, a DVD player or both in their home. When asked, 30 students answered yes to "Is there a VCR in your home?", and 20 students answered yes to "Is there a DVD player in your home?". If 14 of the students answered yes to both questions, how many students are in the class?

A: 24	B: 30	C: 36	D: 64	E: Cannot be determined.

Solution: Let V be the set of all students who have a VCR and D be the set of all students who have a DVD. We are told that $n(V) = 30$, $n(D) = 20$ and $n(V \cap D) = 14$. As well, we are told that all students have one or both of these appliances, which means that $V \cup D = U$, where U is the set of all students in the class. Thus we see that

$$n(U) = n(V \cup D) = n(V) + n(D) - n(V \cap D) = 30 + 20 - 14 = 36$$

There are 36 students in the class. Answer: **C**

2. A certain office building contains 100 offices. There are 20 large offices, 35 medium-sized offices and 45 small offices. All of the large offices have windows, but only 15 of the small offices have windows. If 60 of the offices in the building have windows, how many of the medium-sized offices do not have a window?

A: 0	B: 10	C: 25	D: 30	E: Cannot be determined.

Solution: Let L be the set of large offices, M be the set of medium-sized offices and S be the set of small offices. Also, let W be the set of all offices which have windows. We are told that $n(U) = 100$, $n(L) = 20$, $n(M) = 35$ and $n(S) = 45$. (Notice that $n(L) + n(M) + n(S) = n(U)$, so there are not other kinds of offices.) We also know that $n(L \cap W) = n(L) = 20$ and that $n(S \cap W) = 15$, as well as that $n(W) = 60$. Since $n(W) = n(L \cap W) + n(M \cap W) + n(S \cap W)$, we see that

$$n(M \cap W) = n(W) - [n(L \cap W) + n(S \cap W)] = 60 - (20 + 15) = 25$$

Finally, since $n(M) = n(M \cap W) + n(M \cap W^c)$, we get

$$n(M \cap W^c) = n(M) - n(M \cap W) = 35 - 25 = 10$$

Answer: **B**

3. How many (positive) even integers less than 5000 can be formed using only digits which are in the set $\{1, 2, 3, 4\}$, if repetition is allowed?

A: 12	B: 32	C: 128	D: 170	E: 340

Solution: We must realize here that integers of differing lengths can be made, and consider each possible length as a separate case. Since the integers we're forming must be less than 5000, they can have at most 4 digits.

Case 1: 1-digit integers

Since the set contains 2 even digits, there are 2 even one-digit integers that can be formed.

Case 2: 2-digit integers
Since 0 is not one of the digits in the set, the first digit may contain any of the 4 digits in the set. The second must be one of the 2 even digits. Therefore there are $4 \times 2 = 8$ even 2-digit integers that can be formed.
Case 3: 3-digit integers
Each of the first 2 digits can be any of 1, 2, 3 or 4, but the last must be either 2 or 4. This gives $4 \times 4 \times 2 = 32$ even 3-digit integers that can be formed.
Case 4: 4-digit integers
Since all of the available digits are less than 5, so that all 4-digit integers which can be formed are less that 5000, then any of the 4 available digits can be used for each of the first 3 positions of a 4-digit integer. Of course, the last digit can still only be 2 or 4. There are $4 \times 4 \times 4 \times 2 = 128$ even 4-digit integers which can be formed.
Total: There are $2 + 8 + 32 + 128 = 170$ even integers under 5000 which can be formed. Answer: **D**

4. Each student in a certain class must write an essay. The professor has provided a list of 6 acceptable essay topics. (Each student must choose one of these topics for his or her essay.) If there are 15 students in the class, in how many different ways can the students choose topics for their essays?

A: $15!$	B: $\frac{15!}{9!}$	C: $\binom{15}{6}$	D: 15^6	E: 6^{15}

Solution: Each of the 15 students must decide which one of the 6 topics to choose. This gives 15 decisions, with 6 choices available for each, so there are 6^{15} different ways the students can choose essay topics. Answer: **E**

5. Ted is having dinner at a restaurant and has ordered ice cream for dessert. The waitress says there are 5 flavours of ice cream available. As well, she asks whether Ted would like any one topping (strawberries, chocolate sauce or butterscotch), and/or whipped cream on his ice cream. Finally, she asks whether he wants chocolate sprinkles. In how many different ways could Ted order his ice cream?

A: 13	B: 15	C: 32	D: 60	E: 80

Solution: We know that Ted is going to have ice cream. The 5 flavours give 5 choices for Ted's decision of "which flavour of ice cream will I have?". Ted also has to decide "which topping, if any, shall I have on my ice cream?", with 4 choices (3 toppings plus the 'no topping' choice) available. Next, Ted needs to decide "will I have whipped cream on it?" (yes or no, so 2 choices). Finally, Ted decides "do I want chocolate sprinkles?" (again 2 choices). Altogether there are

$$5 \times 4 \times 2 \times 2 = 80$$

different ways that Ted could order his ice cream. Answer: **E**

6. Using the letters $a, b, c, d, e, f, g, h, i$ and j, how many 3-letter passwords can be formed if repetition is not allowed?

A: 1000	B: 729	C: 720	D: 504	E: 120

Solution: There are 10 letters available for use. If repetition is not allowed, then each possible password is an arrangement of 3 different letters. There are

$$\binom{10}{3} \times 3! \text{ or } \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \times 9 \times 8 = 720$$

different passwords which can be formed. Answer: **C**

7. Refer to question 6. How many 3-letter passwords can be formed if repetition *is* allowed?

A: 1000	B: 729	C: 720	D: 504	E: 120

Solution: If repetition is allowed, then all 10 letters are available for each of 3 decisions (positions in the password), so in this case there are

$$10 \times 10 \times 10 = 1000$$

different passwords which can be formed. Answer: **A**

8. How many subsets of the digits 0 through 9 are there?

A: 2^9	B: 2^{10}	C: 9^2	D: 10^2	E: $10!$

Solution: We know that a set containing n elements has 2^n subsets. Thus the set containing the 10 digits from 0 through 9 has 2^{10} different subsets. Answer: **B**

9. How many subsets of 3 of the digits 0 through 9 are there?

A: 1000	B: 729	C: 720	D: 504	E: 120

Solution: A set of n objects has $\binom{n}{k}$ different subsets of size k. Thus there are

$$\binom{10}{3} = \frac{10!}{3!7!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$$

different subsets of 3 of these 10 digits. Answer: **E**

10. In how many different ways can the letters of the word 'oranges' be arranged?

A: 7	B: $7!$	C: 2^7	D: 7^7	E: $\binom{26}{7}$

Solution: This word contains 7 letters, all distinct, so there are 7! different ways the letters can be arranged. Answer: **B**

11. In how many different ways can the letters of the word 'bananas' be arranged?

A: $7!$	B: 2^7	C: $\binom{7}{3} \times \binom{7}{2}$	D: $\frac{7!}{3!2!}$	E: $\frac{7!}{3!2!} \times \frac{1}{2!}$

Solution: This time the 7 letters are not all distinct. There are 3 A's, 2 N's, 1 B and 1 S, so there are

$$\binom{7}{3\ 2\ 1\ 1} = \frac{7!}{3!2!1!1!} = \frac{7!}{3!2!}$$

different arrangements of the letters. Answer: **D**

12. Terry is packing lunches for 6 different children. There are 3 apples, 2 bananas and 1 orange. In how many distinct ways can Terry put 1 piece of fruit in each child's lunch? (Assume fruit of the same type are indistinguishable, so only the type of fruit matters.)

A: 12	B: 24	C: 41	D: 60	E: 1800

Solution: Terry must divide the 6 children up into groups of 3 to receive apples, 2 to receive bananas and 1 to receive an orange. There are

$$\binom{6}{3\ 2\ 1} = \frac{6!}{3!2!1!} = \frac{6\times5\times4\times3!}{3!2!} = 60$$

different ways to distribute the fruit. Answer: **D**

13. There are 6 different books which Gary and Mary must read to research a project they're working on together. In how many different ways can they divide up the books between them if each is going to read 3 books?

A: $\binom{6}{3\ 3}$	B: $\binom{6}{3\ 3} \div 2!$	C: $\binom{6}{3} + \binom{3}{3}$	D: 2^6	E: $6!$

Solution: They need to divide the 6 distinct books into 2 distinguishable groups of 3 – Gary's group and Mary's group. There are $\binom{6}{3\ 3}$ different ways in which this can be done. Answer: **A**

14. There are 6 different books to be placed in 2 identical carrier bags. In how many distinct ways can the books be divided up to be put in the bags if 3 books are to be put in each bag?

A: $\binom{6}{3\ 3}$	B: $\binom{6}{3\ 3} \div 2!$	C: $\binom{7}{2}$	D: $\binom{7}{6}$	E: 2^6

Solution: This time, since the 2 bags are identical, the 6 distinct books are to be divided up into 2 indistinguishable groups of 3. There are $\binom{6}{3\ 3} \div 2!$ different ways in which this can be done. Answer: **B**

15. There are 6 identical copies of the same book to be placed in 2 identical carrier bags. In how many distinct ways can the books be divided up to be put in the bags if 3 books are to be put in each bag?

A: $\binom{6}{3\ 3}$	B: $\binom{6}{3\ 3} \div 2!$	C: $\binom{7}{6}$	D: 2^6	E: 1

Solution: This time, the 6 books are all identical. There is only 1 way to divide 6 identical objects up into 2 groups of 3. Answer: **E**

16. The London Public Library has just received 40 (identical) copies of a recent best selling book. In how many distinct ways can these books be distributed among the 15 different library branches if each branch must receive at least 1 copy of the book?

A: $\binom{54}{40}$	B: $\binom{54}{15}$	C: $\binom{39}{25}$	D: $\binom{39}{15}$	E: 15^{40}

Solution: Since the books are identical, we are dealing with a free distributions problem. There are 40 copies of the book, but we know that each of the 15 libraries must receive at least 1. Allocating one to each library accounts for 15 of the 40 books, leaving 25 still to be allocated. These may each be given to any library, so we have $k = 25$ identical books to distribute among $r = 15$ libraries. The number of ways in which this can be done is $\binom{k+r-1}{k} = \binom{25+15-1}{25} = \binom{39}{25}$. Answer: **C**

17. Every customer who orders an extra large pizza today from Pepe's Perfect Pizza will be offered the anniversary special: they can either get a small pizza (along with their extra large) for an extra \$2, or they can get a second extra large pizza for an extra \$5. They cannot get both, and some customers, of course, may not want either. Pepe wants to know how many different possibilities there are for having to provide these special pizzas, assuming that 50 people order extra large pizzas today (i.e. 50 people are

offered this special). Pepe does not care about which customers order the small or extra large special. He just cares about how many customers order each. (e.g. 50 take the small special, or 20 take the small special, 20 take the XL special and 10 take neither.) How many different possibilities are there?

A: 3^{50}	B: 50^3	C: $\binom{52}{50}$	D: $\binom{52}{3}$	E: 50×3

Solution: Since Pepe doesn't care about which customers order what, but only how many, we treat the customers as identical. The 50 customers may each choose any of 3 specials: small, another extra large or neither. Thus we want to know the number of ways in which $k = 50$ identical customers could be divided into $r = 3$ distinct categories, which is $\binom{50+3-1}{50} = \binom{52}{50}$. Answer: **C**

18. $S = \{a, b, c\}$ is a sample space for a certain experiment. If $\Pr[c] = 2\Pr[b]$ and $\Pr[b] = 3\Pr[a]$, what is the value of $\Pr[a]$?

A: $\frac{1}{10}$	B: $\frac{1}{6}$	C: $\frac{1}{3}$	D: $\frac{6}{10}$	E: Cannot be determined.

Solution: Since S is a sample space, we know that

$$Pr[S] = Pr[a] + Pr[b] + Pr[c] = 1$$

We have $Pr[b] = 3Pr[a]$ and $Pr[c] = 2Pr[b] = 6Pr[a]$ so that

$$Pr[a] + Pr[b] + Pr[c] = Pr[a] + 3Pr[a] + 6Pr[a] = 10Pr[a]$$

Therefore $10Pr[a] = 1$, so $Pr[a] = \frac{1}{10}$. Answer: **A**

19. Six children and one teacher sit in a circle to play a game. If they arrange themselves at random in the circle, what is the probability that Mary is sitting beside the teacher?

A: $\frac{1}{6}$	B: $\frac{1}{3}$	C: $\frac{2}{7}$	D: $\frac{2}{6!}$	E: $\frac{1}{15}$

Solution: Letting S be the set of all ways in which the 7 people can sit in a circle, we have $n(S) = (7-1)! = 6!$. Let E be the event that Mary is sitting beside the teacher. We can think of the teacher and 5 of the children (everyone except Mary) arranging themselves in the circle, which can be done in $(6-1)! = 5!$ different ways. Now, we have Mary take her place in the circle, in one of the 2 positions beside the teacher (i.e. to the left or to the right of the teacher). This gives $n(E) = 5! \times 2$ different ways to form a circle with Mary sitting beside the teacher. Thus when the 7 people arrange themselves at random in the circle, the probability that Mary will be sitting beside the teacher is

$$Pr[E] = \frac{n(E)}{n(S)} = \frac{5! \times 2}{6!} = \frac{2}{6} = \frac{1}{3}$$

Answer: **B**

20. A 3 card hand is dealt at random from a standard deck of cards. What is the probability that the hand contains at least 2 clubs?

A: $\frac{\binom{13}{2}\binom{39}{1} + \binom{13}{3}}{\binom{52}{3}}$	B: $\frac{\binom{13}{2} + \binom{13}{3}}{\binom{52}{3}}$	C: $\frac{\binom{13}{2}\binom{39}{1}}{\binom{52}{3}}$	D: $\frac{\binom{13}{2}\binom{50}{1}}{\binom{52}{3}}$	E: $\frac{13 \times 13 \times 39}{(52)^3}$

Solution: Let S be the set of all 3-card hands which could be dealt. Then $n(S) = \binom{52}{3}$. Let E be the event that the hand contains at least 2 clubs. Then E occurs whenever the hand contains either 2 or 3 clubs. There are $\binom{13}{2}\binom{39}{1}$ different 3-card hands which contain 2 clubs and one other (non-club) card. Also, there are $\binom{13}{3}$ different 3-card hands which contain 3 clubs. Therefore $n(E) = \binom{13}{2}\binom{39}{1} + \binom{13}{3}$ so that $Pr[E] = \frac{\binom{13}{2}\binom{39}{1}+\binom{13}{3}}{\binom{52}{3}}$. Answer: **A**

21. A 3 card hand is dealt at random from a standard deck of cards. If it is known that the hand contains *exactly* 2 clubs, what is the probability that the hand contains exactly 1 diamond?

A: 0	B: $\frac{1}{4}$	C: approx. .3047	D: $\frac{1}{3}$	E: 1

Solution: Let S be defined as in the previous question. Let F be the event that a 3-card hand contains exactly 2 clubs and G be the event that a 3-card hand contains exactly 1 diamond. Then we are looking for $Pr[G|F] = \frac{Pr[G \cap F]}{Pr[F]} = \frac{n(G \cap F)}{n(F)}$ (since S is an equiprobable sample space). Therefore we need to find $n(G \cap F)$ and $n(F)$. Of course $n(F)$ is just the number of 3-card hands which contain exactly 2 clubs which (as we saw in the previous question) is given by $n(F) = \binom{13}{2}\binom{39}{1}$. And $n(G \cap F)$ is the number of hands which contain exactly 2 clubs and also contain exactly 1 diamond, which is given by $n(G \cap F) = \binom{13}{2}\binom{13}{1}$. Thus we get

$$Pr[G|F] = \frac{n(G \cap F)}{n(F)} = \frac{\binom{13}{2}\binom{13}{1}}{\binom{13}{2}\binom{39}{1}} = \frac{13}{39} = \frac{1}{3}$$

Answer: **D**

22. A single card is drawn from a standard deck. Let H be the event that a Heart is drawn and E be the event that the card is not a spade. Find $\Pr[H \mid E]$.

A: $\frac{\binom{13}{1}\binom{39}{1}}{\binom{52}{2}}$	B: $\frac{3}{4}$	C: $\frac{1}{2}$	D: $\frac{1}{4}$	E: $\frac{1}{3}$

Solution: We know that there are $n(S) = 52$ ways to draw a single card from a deck. We have $n(H) = 13$ hearts in the deck and $n(E) = 39$ non-spades in the deck. To find $Pr[H|E] = \frac{n(H \cap E)}{n(E)}$, we need to realize that *all* of the hearts are not spades, so that $n(H \cap E) = n(H) = 13$. This gives $Pr[H|E] = \frac{13}{39} = \frac{1}{3}$. That is, since hearts make up one-third of the cards which are not spades, then the probability that a card which is not a spade is a heart is one-third. Answer: **E**

23. Refer to question 22. Are H and E independent events?

A: Yes	B: No	C: Cannot be determined.

Solution: No. Since $Pr[H \cap E] = Pr[H] = \frac{13}{52} = \frac{1}{4}$, while $Pr[E] = \frac{39}{52} = \frac{3}{4}$, then

$$Pr[H] \times Pr[E] = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16} \neq Pr[H \cap E]$$

Answer: **B**

24. There are 3 identical-looking boxes. One box contains a triangle, a square and a circle. The second box contains a triangle and an oval. The third box contains one square and 2 triangles. One box is selected at random and a single shape is chosen from that box. What is the probability that the chosen shape is an oval?

A: $\frac{1}{8}$	B: $\frac{1}{6}$	C: $\frac{1}{4}$	D: $\frac{1}{3}$	E: $\frac{1}{2}$

Solution: Let B_1, B_2 and B_3 denote the 3 boxes, in the order mentioned above. Also, let T, S, C, and O denote, respectively, that a triangle, a square, a circle or an oval is chosen when one shape is drawn from a box. Since only one of the boxes, B_2, contains an oval, event O can only occur when box B_2 is chosen. That is, we have $Pr[O] = Pr[B_2 \cap O]$. Of course, we also know that $Pr[B_2 \cap O] = Pr[O|B_2] \times Pr[B_2]$.

Since one box is selected at random, each box is equally likely to be chosen, so $Pr[B_2] = \frac{1}{3}$. And since B_2 contains two shapes, only one of which is an oval, then $Pr[O|B_2] = \frac{1}{2}$. Therefore we have

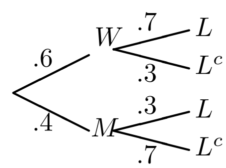
$$Pr[O] = Pr[O \cap B_2] = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

Answer: **B**

25. In a large class, 60% of the students are women. It is known that 70% of the women and 30% of the men in this class have long (at least shoulder length) hair. A single student is chosen at random from the class. What is the probability that the student has long hair?

A: .3	B: .42	C: .54	D: .7	E: 1

Solution: Let W be the event that a student is a woman and M be the event that a student is a man. Also, let L be the event that a student has long hair. We can use a probability tree to find $Pr[L]$.



We get $Pr[L]$ by adding up the path probabilities for all paths containing an L branch. We get

$$Pr[L] = (.6)(.7) + (.4)(.3) = .42 + .12 = .54$$

Answer: **C**

26. Refer to question 25. If the chosen student does have long hair, what is the probability that the student is a man?

A: $\frac{2}{9}$	B: $\frac{2}{7}$	C: $\frac{3}{25}$	D: $\frac{3}{10}$	E: $\frac{27}{125}$

Solution: Here, we could use Bayes' theorem. However, since we have already found $Pr[L]$ there's really no need. We want to know $Pr[M|L]$. We have

$$Pr[M|L] = \frac{Pr[M \cap L]}{Pr[L]} = \frac{(.4)(.3)}{.54} = \frac{.12}{.54} = \frac{12}{54} = \frac{2(6)}{9(6)} = \frac{2}{9}$$

Answer: **A**

Use the following information for questions 27, 28 and 29.

10 standard decks of cards are shuffled (separately) and a single card is drawn from each deck.

27. What is the probability that exactly 1 Heart is drawn?

A: $\left(\frac{1}{4}\right)^{10}$	B: $\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{9}$	C: $10 \times \left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{9}$	D: $\frac{\binom{520}{130}}{\binom{520}{10}}$	E: $\frac{\binom{130}{1}\binom{390}{9}}{\binom{520}{10}}$

Solution: $n = 10$ independent trials of the experiment "draw a card from a standard deck" are being performed. If we define success to be that a Heart is drawn, we have $p = \frac{1}{4}$. The probability that exactly $k = 1$ success is observed on the 10 trials is

$$Pr[\text{exactly } k = 1 \text{ success}] = \binom{n}{k} p^n (1-p)^{n-k} = \binom{10}{1} \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^9$$

And of course $\binom{10}{1} = 10$. Answer: **C**

28. What is the probability that at least 1 Heart is drawn?

A: $10 \times \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^9$	B: $1 - 10 \times \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^9$	C: $\left(\frac{3}{4}\right)^{10}$
D: $1 - \left(\frac{3}{4}\right)^{10}$	E: $\left(\frac{1}{4}\right)^{1+2+\cdots+10}$	

Solution: Observing at least one success is the complement of observing no successes. We define success as above and get

$$Pr[\text{no successes}] = \binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10} = 1 \times 1 \times \left(\frac{3}{4}\right)^{10} = \left(\frac{3}{4}\right)^{10}$$

which gives $Pr[\text{at least one success}] = 1 - Pr[\text{no successes}] = 1 - \left(\frac{3}{4}\right)^{10}$. Answer: **D**

29. (Refer to previous information.) What is the probability that 2 Hearts, 3 Spades, 4 Clubs and 1 Diamond are drawn?

A: $\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^3\left(\frac{1}{4}\right)^4\left(\frac{1}{4}\right)^1$
B: $\frac{10!}{2!3!4!1!}\left(\frac{1}{4}\right)^{10}$
C: $\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^1$
D: $\binom{10}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^8 + \binom{10}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^7 + \binom{10}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^6 + \binom{10}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^9$
E: $10! \times \left(\frac{1}{4}\right)^{10}$

Solution: Let H, S, C and D represent, respectively, the events that a Heart, a Spade, a Club or a Diamond is drawn, when one card is drawn from a standard deck. The the probably of getting each of these kinds of cards is the same for all, i.e., $p_H = p_S = p_C = p_D = \frac{1}{4}$. We wish to find the probability that event H occurs $n_H = 2$ times, event S occurs $n_S = 3$ times, event C occurs $n_C = 4$ times and event D occurs $n_D =$ one time, when $n = 10$ trials are performed. We get

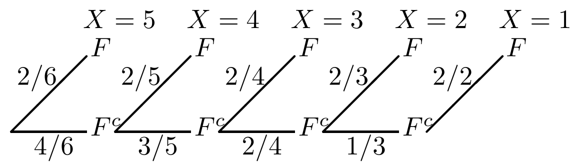
$$\binom{n}{n_H\ n_S\ n_C\ n_D}(p_H)^{n_H}(p_S)^{n_S}(p_C)^{n_C}(p_D)^{n_D} = \binom{10}{2\ 3\ 4\ 1}\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^3\left(\frac{1}{4}\right)^4\left(\frac{1}{4}\right)^1 = \frac{10!}{2!4!3!1!}\left(\frac{1}{4}\right)^{10}$$

Answer: **B**

30. I have 6 loose keys in a box. Two of these keys are spare keys for my front door. The others are keys to neighbours' and friends' houses. The keys all look similar, so I can't tell which ones will unlock my front door. I need to find a spare key for my house, so I choose keys randomly from the box (without replacement) and try them in my front door until I find one that works. Let X be the number of keys left in the box when I find one that works. What is $\Pr[X \geq 3]$?

A: $\frac{1}{5}$	B: $\frac{2}{5}$	C: $\frac{3}{5}$	D: $\frac{4}{5}$	E: 1

Solution: Let F be the event that a selected key unlocks my front door. A probability tree will help in analyzing this situation. The keys are tried without replacement, so the number of keys in the box, and likewise the probability of selecting one that unlocks my front door, changes on each draw. Selection of keys stops as soon as I find one that works, so each path in the tree ends as soon as it contains an F branch. We can put the value of X corresponding to a path through the tree at the end of the path. Since X is the number of keys remaining in the box after I find one that works, and since each key removed from the box corresponds to a branch in the path, then the value of X for any path is given by 6 minus the number of branches in the path.



To find $Pr[X \geq 3]$, we add up the path probabilities for all paths in which the corresponding X-value is at least 3. We get

$$Pr[X \geq 3] = \frac{1}{3} + \frac{2}{3}\left(\frac{2}{5}\right) + \frac{2}{3}\left(\frac{3}{5}\right)\left(\frac{1}{2}\right) = \frac{1}{3} + \frac{4}{15} + \frac{1}{5} = \frac{5+4+3}{15} = \frac{12}{15} = \frac{4}{5}$$

Answer: **D**

31. Let X be a discrete random variable with $\Pr[X=-1] = \frac{1}{3}, \Pr[X=0] = \frac{1}{6}$ and $\Pr[X=1] = \frac{1}{2}$. What is the mean of X?

A: 0	B: $\frac{1}{36}$	C: $\frac{1}{6}$	D: $\frac{5}{6}$	E: Cannot be determined.

Solution:

$$\mu = (-1)\left(\frac{1}{3}\right) + (0)\left(\frac{1}{6}\right) + (1)\left(\frac{1}{2}\right) = -\frac{1}{3} + 0 + \frac{1}{2} = \frac{1}{6}$$

Answer: **C**

32. Discrete random variable X has $\Pr[X=1] = \frac{1}{4}$ and $\Pr[X=2] = \frac{3}{4}$, so that $\mu = \frac{7}{4}$. Find $V(X)$.

A: $\frac{7}{4}$	B: $\frac{49}{16}$	C: $\frac{13}{4}$	D: $\frac{3}{16}$	E: $\frac{\sqrt{3}}{4}$

Solution: We see that

$$E(X^2) = (1)^2\left(\frac{1}{4}\right) + (2^2)\left(\frac{3}{4}\right) = \frac{1}{4} + \frac{12}{4} = \frac{13}{4}$$

so we have

$$V(X) = E(X^2) - \mu^2 = \frac{13}{4} - \left(\frac{7}{4}\right)^2 = \frac{52}{16} - \frac{49}{16} = \frac{3}{16}$$

Answer: **D**

33. Let X be a discrete random variable with mean $\mu = 5$ and variance $V(X) = 2$. What is the standard deviation of X?

A: 27	B: -21	C: 2	D: $\sqrt{21}$	E: $\sqrt{2}$

Solution: The standard deviation is $\sigma(X) = \sqrt{V(X)} = \sqrt{2}$. Answer: **E**

34. Let X be a discrete random variable with mean $\mu = 8$ and variance $V(X) = 36$. What is the value of $E(X^2)$?

A: 100	B: 64	C: 44	D: 14	E: 6

Solution: Since $V(X) = E(X^2) - \mu^2$, then $E(X^2) = V(X) + \mu^2 = 36 + 8^2 = 36 + 64 = 100$. Answer: **A**

Use the following information for questions 35, 36 and 37.

Discrete random variable X has $E(X) = 10$ and $V(X) = 4$. Y is a discrete random variable whose value is given by $Y = 25 - 2X$.

35. Find $E(Y)$.

A: 20	B: -20	C: 5	D: 17	E: Cannot be determined.

Solution: We have $E(Y) = E(25 - 2X) = 25 - 2E(X) = 25 - 2(10) = 5$. Answer: **C**

36. Find $\sigma(Y)$.

A: 4	B: -4	C: 5	D: 16	E: 21

Solution: We get $\sigma(Y) = \sigma(25 - 2X) = \sigma((-2)X + 25) = |-2|\sigma(X) = 2\sqrt{4} = 2(2) = 4$. Answer: **A**

37. Are X and Y independent random variables?

A: Yes	B: No	C: Cannot be determined.

Solution: No. Since $Y = 25 - 2X$, Y depends on X. That is, each value of X corresponds to a single value of Y. Suppose that x_1 and x_2 are 2 possible values of X, with $x_1 \neq x_2$. Let $y_1 = 25 - x_1$ and $y_2 = 25 - 2x_2$ so that y_1 and y_2 are possible values of Y. Then $Pr[(X = x_1) \cap (Y = y_2)] = 0$, since when X has the value x_1, Y has the value y_2. However $Pr[X = x_1] \neq 0$ and $Pr[Y = y_2] \neq 0$, so $Pr[(X = x_1) \cap (Y = y_2)] \neq Pr[X = x_1] \times Pr[Y = y_1]$. Answer: **B**

38. A fair coin is tossed twice. Discrete random variable X has the value 0 if the first toss comes up Heads and otherwise has the value 1. Discrete random variable Y is the number of times the coin comes up Heads during the 2 tosses. Find $V(X + Y)$.

A: $\frac{1}{4}$	B: $\frac{1}{2}$	C: $\frac{3}{4}$	D: $\frac{3}{2}$	E: $\frac{5}{2}$

Solution: The set $S = \{HH, HT, TH, TT\}$, where H denotes that the coin comes up Heads, T denotes that the coin comes up Tails, and the pairs of letters denote the results of the 2 tosses, in order, is an equiprobable sample space for any experiment involving tossing a fair coin twice.

Outcome	HH	HT	TH	TT
X-value	0	0	1	1
Y-value	2	1	1	0
Value of $X + Y$	2	1	2	1

We see that $(X + Y = 1)$ and $(X + Y = 2)$ are equally likely, each occurring with half the outcomes. Therefore

$$E(X + Y) = 1Pr[X + Y = 1] + 2Pr[X + Y = 2] = 1(.5) + 2(.5) = 1.5$$

and $E[(X+Y)^2] = (1)^2(.5) + (2)^2(.5) = 2.5$ so that

$$V(X+Y) = 2.5 - (1.5)^2 = 2.5 - 2.25 = .25$$

Answer: **A**

39. A fair coin is tossed 10 times. Let X be the number of times Tails comes up. Which of the following statements is **false**?

A: $E(X) = 5$	B: $V(X) = \frac{5}{2}$
C: $\Pr[X=5] = \frac{63}{256}$	D: $\Pr[X \geq 1] = 1 - \frac{1}{2^{10}}$
E: The possible values of X are the integers from 1 to 10.	

Solution: We have $X = B(10, .5)$. That is, X is counting the number of successes in $n = 10$ Bernoulli trials in which success is defined to be that the coint comes up Tails, so that the probability of success is $p = .5$. Therefore $E(X) = np = 10(.5) = 5$ and $V(X) = np(1-p) = 10(.5)(.5) = 2.5$. Also,

$$Pr[X=5] = \binom{10}{5}(.5)^5(.5)^5 = \frac{10!}{5!5!}\left(\frac{1}{2}\right)^{10} = \frac{10(9)(8)(7)(6)}{5(4)(3)(2)(1)} \times \frac{1}{2^{10}} = \frac{2(3)(2)(7)(3)}{2^{10}} = \frac{63}{2^8} = \frac{63}{256}$$

As well, since $Pr[X \geq 1] = 1 - Pr[X=0]$ we have

$$Pr[X \geq 1] = 1 - \binom{10}{0}(.5)^0(.5)^{10} = 1 - \left(\frac{1}{2}\right)^{10} = 1 - \frac{1}{2^{10}}$$

We see that statements A through D are all true. However, there could be no successes in the 10 trials, so the possible values of X are the integers from 0 to 10, not just from 1 to 10, and we see that statement E is not true. Answer: **E**

Use the following information for questions 40 and 41.

A single card is drawn from a well-shuffled deck. Discrete random variable X has the value 1 if the card is black and 3 if the card is red. Discrete random variable Y has the value 1 when the card is a 2,3,4,5,6,7,8,9 or 10 and has the value 5 when the card is a Jack, Queen, King or Ace. Values of $\Pr[(X=x) \cap (Y=y)]$ are shown in the table below:

	$(Y=1)$	$(Y=5)$
$(X=1)$	18/52	8/52
$(X=3)$	18/52	8/52

It can easily be verified that $E(X) = 2$ and $\sigma(X) = 1$, while $E(Y) = \frac{29}{13}$ and $\sigma(Y) = \frac{24}{13}$.

40. Are X and Y independent random variables?

A: Yes	B: No	C: Cannot be determined.

Solution: We have $Pr[X=1] = Pr[X=3] = \frac{1}{2}$ as well as $Pr[Y=1] = \frac{9}{13}$ and $Pr[Y=5] = \frac{4}{13}$. Therefore we have

$$Pr[X=1] \times Pr[Y=1] = \frac{1}{2} \times \frac{9}{13} = \frac{9}{26} = \frac{18}{52}$$

Likewise,

$$Pr[X=1] \times Pr[Y=5] = \frac{1}{2} \times \frac{4}{13} = \frac{8}{52}$$

For $Pr[X=3] \times Pr[Y=1]$ and $Pr[X=3] \times Pr[Y=5]$ we do exactly the same arithmetic. Therefore we see that in every case,

$$Pr[(X=x) \cap (Y=y)] = Pr[X=x] \times Pr[Y=y]$$

so that X and Y **are** independent random variables. Answer: **A**

41. (Refer to above.) You pay \$5 to play a game in which you will draw a card and receive \$W, where $W = XY$. What is the expected value of your net winnings from playing this game? (Answer to the nearest cent.)

A: \$9.46	B: \$4.46	C: \$1.85	D: -\$0.54	E: -\$1.59

Solution: The \$5 cost to play the game must be subtracted from the amount you receive in calculating your net winnings. Thus

$$E(W) = E(XY - 5) = E(XY) - 5$$

Since X and Y are independent random variables, then we know that

$$E(XY) = E(X) \times E(Y) = 2 \times \frac{29}{13} = \frac{58}{13}$$

This gives

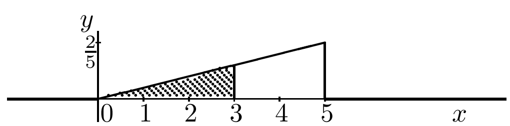
$$E(W) = \frac{58}{13} - 5 = \frac{58}{13} - \frac{65}{13} = -\frac{7}{13}$$

and since this is measured in dollars, we see that $E(W) \approx -\$0.54$. Answer: **D**

42. Continuous random variable X has probabilitiy density function $f(x) = \frac{2x}{25}$ if $0 \le x \le 5$ and $f(x) = 0$ otherwise. Find $\Pr[X < 3]$.

A: $\frac{4}{25}$	B: $\frac{6}{25}$	C: $\frac{9}{25}$	D: $\frac{16}{25}$	E: $\frac{1}{4}$

Solution: To find $Pr[X < 3]$, we need to find the area under the density function from $x = -\infty$ to $x = 3$. Of course, since $f(x) = 0$ for $x < 0$, then there is no area under the density function from $x = -\infty$ to $x = 0$, so we need to find the area under this function from $x = 0$ to $x = 3$.
The region described by this density function is triangular, with $f(0) = 0$ and $f(5) = \frac{2}{5}$ so that the line segment providing the top edge of the triangle runs from the point $(0, 0)$ to the point $(5, 2/5)$. We need to find the area of the smaller triangle which runs from $x = 0$ to $x = 3$.



The region is a triangle with base 3 and height $f(3) = \frac{6}{25}$, so we get

$$Pr[X < 3] = \frac{1}{2} \times 3 \times \frac{6}{25} = \frac{9}{25}$$

Answer: **C**

43. Continuous random variable X has probability density function $f(x) = cx$ if $0 \le x \le 3$ and $f(x) = 0$ otherwise. What is the value of c?

A: $\frac{9}{2}$	B: 3	C: $\frac{1}{3}$	D: $\frac{2}{9}$	E: Cannot be determined.

Solution: We know that the total area under the density function must be 1. The function $y = cx$ is a straight line with slope c, so the region under the density function is a triangle whose top edge is the segment of this line running from the point $(0, f(0)) = (0, 0)$ to the point $(3, f(3)) = (3, 3c)$. This triangle has base 3 and height $f(3) = 3c$, so we have

$$\frac{1}{2} \times 3 \times 3c = 1 \Rightarrow \frac{9}{2}c = 1 \Rightarrow c = \frac{2}{9}$$

Answer: **D**

44. Discrete random variable X has possible values 2,4,6 and 8. Continuous random variable Y is a good approximation for X. Which of the following approximates $\Pr[X \leq 6]$?

A: $\Pr[5.5 < Y < 6.5]$	B: $\Pr[5 < Y < 7]$	C: $\Pr[Y < 6]$
D: $\Pr[Y < 6.5]$	E: $\Pr[Y < 7]$	

Solution: The possible values of the discrete r.v. X are evenly spaced $k = 2$ units apart, so when we approximate X using the continuous r.v. Y, we must apply a continuity correction of $\frac{k}{2} = 1$. Therefore we approximate the event $(X \leq 6)$ using the event $(Y < 6 + 1) = (Y < 7)$, so $Pr[X \leq 6] = Pr[Y < 7]$. Answer: **E**

45. Discrete random variable X has as its possible values the integers 1 through 10. Continuous random variable Y is a good approximation for X. Which of the following approximates $\Pr[3 < X < 8]$?

A: $\Pr[3 < Y < 8]$	B: $\Pr[3.5 < Y < 7.5]$	C: $\Pr[4 < Y < 7]$
D: $\Pr[2.5 < Y < 8.5]$	E: $\Pr[Y < 7.5] - \Pr[Y < 2.5]$	

Solution: Since discrete r.v. X has possible values which are consecutive integers, i.e. are evenly spaced $k = 1$ unit apart, we apply a continuity correction of $\frac{k}{2} = .5$. Before we apply this continuity correction, we should put the event in terms of X into the form we are most familiar with. We know that, in general when approximating a discrete r.v. by a continuous r.v., $Pr[X \leq c] = Pr[Y < c + k/2]$ and also $Pr[a \leq X \leq b] = Pr[a - k/2 < Y < b + k/2]$. We need either of these forms. Since the values of X are consecutive integers, we have

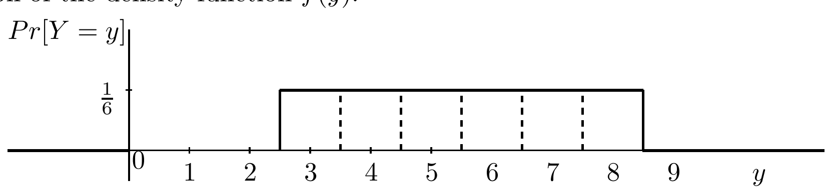
$$(3 < X < 8) = (X > 3) \cap (X < 8) = (X \geq 4) \cap (X \leq 7) = (4 \leq X \leq 7)$$

and we get $Pr[3 < X < 8] = Pr[4 \leq X \leq 7] = Pr[3.5 < Y < 7.5]$. That is, once we have the $Pr[a \leq X \leq b]$ form, we simply extend the interval by $k/2 = .5$ in each direction. Answer: **B**

46. X is a discrete random variable whose possible values are consecutive integers. Continuous random variable Y has probability density function $f(y) = \dfrac{1}{6}$ if $2.5 \leq y \leq 8.5$ and $f(y) = 0$ otherwise. Y is a good approximation for X. Which of the following statements is **false**?

A: The possible values of X are equally likely to occur.
B: The largest possible value of X is 8.
C: X has 6 possible values.
D: $\Pr[X > 3] = \dfrac{5}{6}$
E: $\Pr[3 < X < 5] = \dfrac{1}{2}$

Solution: Since the possible values of X are consecutive integers, they are evenly spaced $k = 1$ unit apart. Thus we know that in using continuous r.v. Y to approximate X, a continuity correction of $\frac{k}{2} = .5$ must be applied. Thus the event $(X = a)$ is approximated by the event $(a - .5 < Y < a + .5)$. That is, the interval used to approximate $(X = a)$ has width 1 and is centred at a.
Consider the graph of the density function $f(y)$.



We can see, from the approximating intervals, that the possible values of X are 3, 4, 5, 6, 7 and 8. Therefore statements B and C are both true. For each of these possible values, the region under the density function in the approximating interval is a rectangle with width 1 and height $\frac{1}{6}$, so $Pr[X = x] = 1 \times \frac{1}{6} = \frac{1}{6}$ for each of the possible values of X. Therefore statement A is true as well. We can calculate

$$Pr[X > 3] = 1 - Pr[X \leq 3] = 1 - Pr[X = 3] = 1 - \frac{1}{6} = \frac{5}{6}$$

and we see that statement D is also true. However we have

$$Pr[3 < X < 5] = Pr[4 \leq X \leq 4] = Pr[X = 4] = \frac{1}{6}$$

so statement E is false. Answer: **E**

In questions 47 through 50, Z is the Standard Normal random variable. The table showing values of the cumulative distribution function for Z is given at the end of the exam.

47. Find $\Pr[Z < 1.75]$.

A: 0.401	B: .0409	C: .9591	D: .9599	E: 1.75

Solution: From the Z-table, $Pr[Z < 1.75] = .9599$. Answer: **D**

48. Find $\Pr[Z < -0.45]$.

A: .1736	B: .3264	C: .45	D: .55	E: .6736

Solution: We use the symmetry of the normal distribution, and complementation:

$$Pr[Z < -0.45] = Pr[Z > 0.45] = 1 - Pr[Z < 0.45] = 1 - .6736 = .3264$$

Answer: **B**

49. Find $\Pr[0.40 < Z < 1.00]$.

A: .60	B: .4967	C: .1859	D: .1554	E: $-.1859$

Solution: We use the fact that $Pr[a < Z < b] = Pr[Z < b] - Pr[Z < a]$.

$$Pr[0.40 < Z < 1.00] = Pr[Z < 1.00] - Pr[Z < 0.40] = .8413 - .6554 = .1859$$

Answer: **C**

50. Find $\Pr[-1.2 < Z < 0.7]$.

A: .9000	B: .8731	C: .6915	D: .6429	E: .1269

Solution: We do the same as in the previous question, but we must use symmetry and complementation to get the final answer. We have $Pr[-1.2 < Z < 0.7] = Pr[Z < 0.7] - Pr[Z < -1.2]$ and we get

$$Pr[Z < -1.2] = Pr[Z > 1.2] = 1 - Pr[Z < 1.2]$$

so we have

$$Pr[-1.2 < Z < 0.7] = Pr[Z < 0.7] - (1 - Pr[Z < 1.2]) = Pr[Z < 0.7] + Pr[Z < 1.2] - 1 = .7580 + .8849 - 1 = .6429$$

Answer: **D**

51. X is a normal random variable with mean $\mu = 50$ and standard deviation $\sigma = 4$. Find $\Pr[X < 55]$.

A: .6217	B: .8708	C: .8944	D: .9162	E: .9938

Solution: When X is a normal r.v. with mean μ and standard deviation σ, $Pr[X < a] = Pr\left[Z < \frac{a-\mu}{\sigma}\right]$ so in this case we get

$$Pr[X < 55] = Pr\left[Z < \frac{55-50}{4}\right] = Pr\left[Z < \frac{5}{4}\right] = Pr[Z < 1.25] = .8944$$

Answer: **C**

52. X is a normal random variable with mean $\mu = 35$ and standard deviation $\sigma = 10$. Find $\Pr[20 < X < 50]$.

A: .8664	B: .8530	C: .8788	D: .1192	E: .999997

Solution:

$$Pr[20 < X < 50] = Pr\left[\frac{20-35}{10} < Z < \frac{50-35}{10}\right] = Pr[-1.5 < Z < 1.5]$$

We can use the fact that $Pr[-k < Z < k] = 2Pr[Z < k] - 1$ (by symmetry) to get

$$Pr[20 < X < 50] = 2Pr[Z < 1.5] - 1 = 2(.9332) - 1 = .8664$$

Answer: **A**

53. The actual weight of a "1 pound" block of butter is normally distributed with mean μ and standard deviation $\sigma = 2$ grams. To the nearest gram, what is the minimum number of grams that the mean weight, μ, should be in order to ensure that there is less than a 1% chance that a "1 pound" block of butter actually contains less than 1 pound, i.e. 454 grams, of butter?

A: 455	B: 456	C: 457	D: 458	E: 459

Solution: Let X be the amount of butter in a "1 pound" block, in grams. Then X is normally distributed with standard deviation $\sigma = 2$ and unknow mean μ. Requiring that there be less than a 1% chance that a block of butter contains less than 454 grams means we need $Pr[X < 454] < .01$ and of course, this means that we require $Pr[X > 454] > .99$. Standardizing this, we have $Pr\left[Z > \frac{454-\mu}{2}\right] > .99$ and recognizing that in order to have more than 99% of the area under the Z-curve to the right of some value, that value must be negative, then we can express the requirement as

$$Pr\left[Z < -\frac{454-\mu}{2}\right] > .99 \Rightarrow Pr\left[Z < \frac{\mu-454}{2}\right] > .99$$

From the Z-table, we see that the smallest value of k for which $Pr[Z < k] > .99$ is $k = 2.33$, so we must have

$$\frac{\mu - 454}{2} = 2.33 \Rightarrow \mu - 454 = 4.66 \Rightarrow \mu = 458.66$$

so expressed to the nearest gram, the minimum value of μ is 459. Answer: **E**

54. The number of paper clips in a box is approximately normally distributed, with mean 100 and standard deviation 3. What is the probability that a box contains fewer than 102 paper clips?

A: .6915	B: .7486	C: .7967	D: .8078	E: .5000

Solution: Let X be the number of paper clips in a box. Then X is a discrete r.v. whose possible values are (consecutive) integers. We know that X is approximately normal, so letting Y be the normal r.v. with the same mean and standard deviation as X, i.e. with $\mu = 100$ and $\sigma = 3$, Y is a good approximation for X. Of course, to approximate X using Y we must apply a continuity correction. Since the possible values of X are evenly spaced $k = 1$ unit apart, the correction we apply is $\frac{k}{2} = .5$. We are asked to find $Pr[X < 102]$.

$$Pr[X < 102] = Pr[X \leq 101] = Pr[Y < 101.5] = Pr\left[Z < \frac{101.5 - 100}{3}\right] = Pr[Z < 0.5] = .6915$$

Answer: **A**

55. According to a recent poll, 36% of the population object to same sex marriage. If 100 people are selected at random, what is the (approximate) probability that no more than 39 of them object to same sex marriage?

A: .5596	B: .6628	C: .6985	D: .7537	E: .7673

Solution: Let X be the number of people in the poll who object to same sex marriage. Then, since 36% of the population object, X is counting the number of successes in $n = 100$ Bernoulli trials in which the probability of success is $p = .36$. That is, when a poll participant is asked 'do you object to same sex marriage', we define success to be that the answer is yes, which will occur with probability .36, and we are performing 100 independent trials of this experiment.
We see that $X = B(100, .36)$. Since $np = 100(.36) = 36 > 5$ and also $n(1-p) = 100(.64) = 64 > 5$ then we know that X is approximately normal. The mean of X is $\mu = np = 36$ and the standard deviation is $\sigma = \sqrt{np(1-p)} = \sqrt{100(.36)(.64)} = 10(.6)(.8) = 4.8$, so if we let Y be the normal r.v. with the same values of μ and σ, Y is a good approximation for X. Of course, since X is a Binomial r.v., so that the possible values of X are consecutive integers, then we must apply a continuity correction of $\frac{k}{2} = .5$.
We are asked to find the probability that no more than 39 successes are observed, i.e. that X is at most 39. We get

$$Pr[X \leq 39] = Pr[Y < 39.5] = Pr\left[Z < \frac{39.5 - 36}{4.8}\right] \approx Pr[Z < 0.73] = .7673$$

Answer: **E**