Everything you need to know for Math 1229 Test 2

Remember, rows go left to right and columns go up and down!

Intro to Systems of Linear Equations

An equation is linear if there's nothing weird happening to the variables (x, y, z)

An equation is not linear if it contains any of the following:

- x^2 (exponents on variables are bad)
- $\frac{1}{x}$ (variables are not allowed in the denominator)
- $\sin x$ (variables inside trig functions are bad)
- $\ln x$ (logarithms of variables are bad)
- e^x (variables in the exponent are bad)
- $\sqrt{x}$ (roots of variables are bad)
- xy (variables being multiplied together are bad)

To put a system of equations in <u>standard form</u> move all the unknowns to the left and all the constants to the right and align the variables vertically.

To put a system of equations into a matrix, first make sure the system is in standard form and then read off the coefficients:

$$\begin{aligned} 2x + y &= 3 \\ 5x - y &= 2 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 5 & -1 & 2 \end{array}\right)$$

A matrix is in row reduced echelon form if:

1. Every row of zeroes that appear in the matrix is at the very bottom
2. The first non-zero entry in each row must be a 1. This is called a <u>leading one</u>.
3. The leading ones form a staircase pattern: the leading one for a row must be further to the left than any leading ones in the rows below it
4. For each leading one, all other entries in it's column must be zero.

Row Reduction

To solve a system of equations, first put the system into standard form. Next, put the system into a matrix. Then, row reduce the matrix.

When putting a matrix into row reduced form, we can perform three types of operations on the rows:

1. Swap two rows
2. Multiply a row by a non-zero constant
3. Replace one row with itself plus a multiple of another row. i.e. R1 -> R1 + 5R3

To turn a number into a leading one, multiply or divide the row by a number

To turn a number into a zero, replace the row with itself plus a multiple of whichever row has a leading one in it

To row reduce a matrix, work column by column. Fix everything in the first column before you move on to the second column.

1. Start by getting a leading 1 in the upper left corner
2. Use that leading one to make everything else in the first column equal to zero
3. Move to the second row, second column. Try to get a 1 in that position.
4. Use that leading one to make everything else below it equal to zero
5. etc..
6. After each leading one has zeroes below it, work from the right to the left to make zeroes above each leading one.

After a system of equations has been row reduced, we have to translate the rows back into equations.

- If any of the equations say something contradictory such as "0 = 1", the system has no solution.
- If the equations are all in the form x = 9, y = 2, z = 4 (each variable equals a number), then the system has a unique solution
- If neither of the above cases occur, then we have infinitely many solutions and we need to identify which variables will become parameters. First, look at your row reduced matrix and circle the leading ones. Any column that contains no circles will need to become a parameter.

Parameters are only needed when you have infinitely many solutions.

Geometrically, ..

no solution means that the objects do not all intersect at once

one solution means the objects intersect in a common point

a one-parameter solution means the objects intersect in a line

a two-parameter solution means the objects intersect in a plane

a three (or more) parameter solution means the objects intersect in a hyperplane

After a system of equations has been put into a matrix, the number of rows of A will equal the number of equations in the system and the number of columns of A will equal the number of unknowns (x, y, z) in the system.

Matrix Operations

Two matrices can be multiplied if the inner dimensions match. If they match, then the outer dimensions will tell us what size the resulting matrix will be.

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{pmatrix}}_{2x3} \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}}_{3x3}$$

These matrices are 2 x 3 and 3 x 3. The inner dimensions are both 3, so we can multiply these and we'll end up with a 2 x 3 matrix.

To multiply matrices, go across the rows in the first matrix and down the columns in the second matrix.

$$\begin{pmatrix}1 & 2 & 3\\4 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\2 & 0 & 1\\1 & 1 & 2\end{pmatrix}=\begin{pmatrix} & & \\ & & \end{pmatrix}$$

If we want the (2, 3) entry in the result, we need to focus on the 2nd row of the first matrix and the 3rd column of the second matrix:

$$\begin{pmatrix} & & \\4 & 0 & 1\end{pmatrix}\begin{pmatrix} & & -1\\ & & 1\\ & & 2\end{pmatrix}=\begin{pmatrix} & & \\ & & \end{pmatrix}$$

The (2, 3) entry will be found in the same we way multiplied vectors earlier:
4 X (-1) + 0 X 1 + 1 X 2 = -4 + 0 + 2 = -2

$$\begin{pmatrix} & & \\4 & 0 & 1\end{pmatrix}\begin{pmatrix} & & -1\\ & & 1\\ & & 2\end{pmatrix}=\begin{pmatrix} & & \\ & & -2\end{pmatrix}$$

The remaining entries can be found in a similar way.

Two matrices can be added if the have the same size (I.e. Same number of rows and same number of columns). In order to add matrices, simply add the corresponding entries.

To find the transpose of a matrix, take the first row and turn it into the first column, take the second row and turn it into the second column, ... Transposing a matrix is like flipping it and the number of rows and columns will end up switching (i.e. A 2 x 3 matrix will become a 3 x 2 matrix after being transposed)

The identity matrix is a square matrix with ones on the diagonal and zeroes everywhere else. If a matrix is multiplied by the identity, the result will be the original matrix. (Multiplying by the identity is like multiplying by the number 1). The following are a few examples of identity matrices:

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$$

When a system of equations is written as Ax = b, A represents the coefficient matrix and b represents the constants that appear on the right side of the equals sign.

Inverse Matrices

The inverse of a matrix is a related matrix that multiplies with the original to give the identity matrix. The inverse matrix is written A^{-1}

Only square matrices have inverses!

To find a 2 x 2 inverse, use the formula: $\begin{pmatrix}a & b\\c & d\end{pmatrix}^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$

(a and d swap positions, b and c swap signs)

To find the inverse of a matrix that's bigger than 2 x 2, first write the matrix together with the identity matrix beside it and row reduce. If you row reduce and end up with the identity on the left, then what you have on the right is the inverse matrix. If you row reduce and do not end up with the identity on the left, then the matrix has no inverse.

$[\ A \mid I\ ] \Rightarrow [\ I \mid A^{-1}\ ]$

Method of Inverses

If you have a system of equations Ax = b and you know the inverse matrix (A^{-1}), then a solution to the system will be $x = A^{-1}b$

Determinants

To find the determinant of a 2 x 2 matrix:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

To find the determinant of a 3 x 3 matrix:

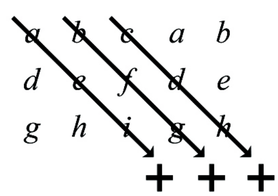
Method 1 (Basketweave method):

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

First, write out the matrix and copy the first two columns and put them to the right of the matrix

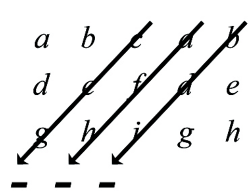
$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$$

Next, multiply along the forward diagonals and add the products



aei + bfg + cdh

Then multiply along the backward diagonals and subtract these:



-ceg-afh-bdi

Combine these together to get your determinant:

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - afh - bdi$$

Important Theorem:

A matrix has no inverse if the determinant is zero!