The first five questions are designed to give you some practice with ideas at the heart of the course, and are of a level with the problems sheets. The last two questions are more challenging, but will give deeper insight into the maths of the course. If you don't have time to do all the questions in the class, you should try to find time to do them before looking at the answers (which will be posted on Moodle).

1. A wave on a string of mass 6 g/m is written as:

$$\psi(z,t) = (270mm) \times \cos\left[2\pi\left(z/(1.2m) + t/(0.2s) + 0.6\right)\right] \tag{1}$$

   (a) What direction does the wave travel ?
   (b) What are the speed, angular frequency, wavelength and wavenumber of the wave ?
   (c) At t=2s, what is the displacement of the wave at z=0.35m ?
   (d) What tension is the string under ?
   (e) What is the maximum displacement for a point on the string ?
   (f) What is the maximum *transverse* speed for a point on the string ?

2. A 1.3m steel wire with mass 0.01 kg, cross-sectional area 1mm^2 and Young's modulus 2×10^{11}Pa is kept under a tension of 800N.

   (a) What is the speed of transverse waves on the wire ? What impedance does the wire present to these waves ?
   (b) If the amplitude of vibration on the wire is 2mm, and the wavelength is the length of the string, what is the maximum kinetic energy per unit length and the maximum power transmitted along the wire for a travelling wave ?

3. A wave travels along a string at a speed of 280 m/s.

   (a) What will be the speed if the string is replaced by one made of the same material and under the same tension but having twice the radius?
   (b) What will be the speed if the string is replaced by one made of a material with half the density and half the radius ?
   (c) What change in the tension will be needed to make the wave travel at a speed of 420 m/s on a string of the original density but half the original radius ?

4. The amplitude of a driven, damped harmonic oscillator ($m\ddot{\psi} + b\dot{\psi} + s\psi = F_0 \cos \omega t$) can be written as:

$$A(\omega) = \frac{F_0}{m}\left(\frac{1}{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}\right)^{\frac{1}{2}} \tag{2}$$

   where $\gamma = b/2m$ as usual.

   (a) Show that the maximum amplitude occurs at a frequency $\omega = \sqrt{\omega_0^2 - 2\gamma^2}$
   (b) What will this frequency be for $\gamma = 0$? What will the amplitude be ? Comment on the physical significance of the result.

5. The frequency of an organ pipe doubles if the length is halved. It is common to combine organ pipes of different lengths to make different effects. What will be the envelope, carrier and beat frequencies if an pipe eight feet long (frequency f) is played at the same time as:

   (a) A pipe $5\frac{1}{3}$ feet long ?
   (b) A pipe four feet long ?
   (c) A pipe $2\frac{2}{3}$ feet long ?
   (d) A pipe $1\frac{3}{5}$ feet long ?

6. Show that the wave equation:

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2} \tag{3}$$

   can be written as:

$$\frac{\partial^2\psi}{\partial u \partial v} = 0 \tag{4}$$

   where $u = x - ct$ and $v = x + ct$. You will need to think about how to write $\partial/\partial x$ and $\partial/\partial t$ in terms of u and v.

7. The damped, driven harmonic oscillator obeys the following equation:

$$m\frac{d^2\psi}{dt^2} = -s\psi - b\frac{d\psi}{dt} + F_0 e^{i\omega t} \tag{5}$$

Show that $\psi(t) = Ae^{i(\omega t+\phi)}$ is a solution to this equation, and that A and ϕ satisfy:

$$Ae^{i\phi} = \frac{F_0}{m}\frac{1}{(\omega_0^2 - \omega^2) + 2i\gamma\omega} \tag{6}$$

$$\tag{7}$$

where $\gamma = b/2m$. Note that A is often written as a magnitude rather than a complex number; you can find this form by evaluating $\sqrt{A(\omega)A^\star(\omega)}$. Now demonstrate, using the expression for A, that the impedance is given by:

$$Z_0(\omega) = b + i(m\omega - s/\omega) \tag{8}$$

1. (a) The wave travels in the $-z$ direction, because the position dependence is z and the kz and ωt parts have the same sign

   (b) The wavenumber $k = 2\pi/1.2 = 5.24\text{m}^{-1}$ and the angular frequency is $\omega = 2\pi/0.2 = 31.42\text{s}^{-1}$. From these, the speed is $\omega/k = 31.42/5.24 = 6.00$m/s. The wavelength is simply $\lambda = 2\pi/k = 1.2$m (though this can be read directly from the form of the wave).

   (c) We have $\psi = 0.27\cos(2\pi \times 0.35/1.2 + 2\pi \times 2/0.2 + 0.6 \times 2\pi) = 0.27 \times (10.89 \times 2\pi) = 0.21$m.

   (d) The tension can be found from the speed and mass per unit length: $c = \sqrt{T/\mu}$ so $T = c^2\mu = 36 * 0.006 = 0.216$ N.

   (e) The maximum displacement will be 0.27m from the amplitude of the wave.

   (f) The maximum transverse velocity is given by the amplitude of $\partial\psi/\partial t = 0.27\omega = 0.27 \times 31.42 = 8.48$m/s.

2. (a) Transverse waves are governed by tension and mass per unit length; we are given $T = 800$N and can find $\mu = 0.01/1.3 = 0.0077$kg/m. This gives a speed of $c = \sqrt{T/\mu} = \sqrt{800/0.0077} = 322.49$m/s (NB if using the rounded value of the mass per unit length, we get 322.33m/s). The impedance is defined as $Z_0 = \sqrt{T\mu} = 2.48$ kg/s.

   (b) If the wavelength is $\lambda = 1.3$m then the wavenumber is $k = 2\pi/1.3 = 4.83\text{m}^{-1}$ and the angular frequency will be $\omega = ck = 322.5 \times 4.83 = 1557.68\text{s}^{-1}$. The kinetic energy per unit length is given by $\frac{1}{2}\mu\dot{\psi}^2$, and the maximum value of $\dot{\psi} = \omega A$. So the maximum kinetic energy is $0.5 \times 0.0077 \times (0.002 \times 1557.68)^2 = 0.037$J. The maximum power transmitted is just twice this multiplied by the velocity, or $0.074 \times 322.5 = 24.1$W (we can see this from the travelling wave equations, or use the formula in the notes where $P = cE = Z_0\dot{\psi}^2$).

3. Assume that the wire had radius R, length L and density ρ. Then $\mu = m/L = \pi R^2 L\rho/L = \pi R^2\rho$.

   (a) The speed is $c = \sqrt{T/\mu}$. If radius doubles, then μ increases by a factor of four, and so the speed will halve to 140 m/s.

   (b) $\mu' = \pi(R/2)^2\rho/2 = \mu/8$. The speed will increase by a factor of $\sqrt{8}$ to 791.96 m/s.

   (c) If the radius is halved then without a change in tension the speed will be doubled. We actually need the speed to increase by a factor of 1.5, so the tension will need to decrease by a factor of 9/16: $c' = 1.5c \Rightarrow T'/\mu' = 2.25T/\mu \Rightarrow T' = 2.25T\mu'/\mu = (9/16)T$.

4. (a) We need to differentiate, and find when $dA/d\omega = 0$.

$$A(\omega) = \frac{F_0}{m}\left((\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2\right)^{-1/2} \quad (1)$$

$$\frac{dA}{d\omega} = \frac{F_0}{m}\left(-\frac{1}{2}\right)\left[2.2\omega.(\omega^2 - \omega_0^2) + 8\gamma^2\omega\right]\left((\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2\right)^{-3/2} \quad (2)$$

$$\Rightarrow 4\omega(\omega^2 - \omega_0^2) + 8\gamma^2\omega = 0 \quad (3)$$

$$\omega^2 - \omega_0^2 = -2\gamma^2 \quad (4)$$

$$\omega^2 = \omega_0^2 - 2\gamma^2 \quad (5)$$

   N.B. We could prove that this was a maximum by considering the second derivative, but the form of the curve makes this clear, and it is not required.

   (b) If $\gamma = 0$ then we have $\omega = \omega_0 = \sqrt{s/m}$. This is an undamped harmonic oscillator, and the amplitude becomes infinite. Clearly this is unphysical: in a real system, non-linear forces will apply giving rise to damping or dissipation of energy somehow.

5. The frequency is inversely proportional to the length.

   (a) The frequency of the second pipe will be $f' = f \times 8/(16/3) = 3f/2$. The envelope and carrier frequencies will be at $f/4$ and $5f/4$ and the beats will have a frequency of $f/2$.

   (b) The second pipe will have $f' = f \times 8/4 = 2f$. The envelope and carrier frequencies will be at $f/2$ and $3f/2$ and the beats will have a frequency of f.

   (c) The second pipe will have $f' = f \times 8/(8/3) = 3f$. The envelope and carrier frequencies will be at f and $2f$ and the beats will have a frequency of $2f$.

   (d) The second pipe will have $f' = f \times 8/(8/5) = 5f$. The envelope and carrier frequencies will be at $2f$ and $3f$ and the beats will have a frequency of $4f$.

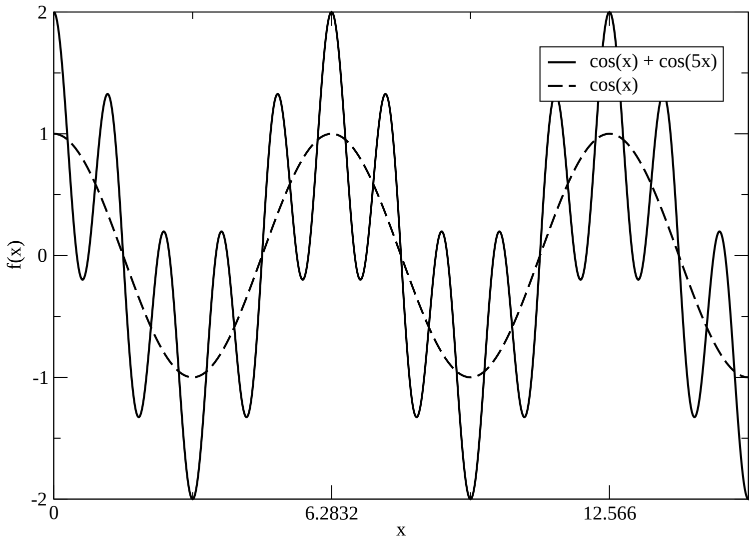

Figure 1: A plot of the wave in 5(d), showing the total (solid line) and the fundamental frequency (dashed line)

Note that for the last two cases, we are no longer in a regime which can really be described as beats; we now have a high frequency signal imposed on a low frequency carrier. The final case is illustrated in Fig. 1.

6. We need lots of differentials:

$$\frac{\partial u}{\partial x} = 1 \tag{6}$$

$$\frac{\partial v}{\partial x} = 1 \tag{7}$$

$$\frac{\partial u}{\partial t} = -c \tag{8}$$

$$\frac{\partial v}{\partial t} = c \tag{9}$$

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \\ &= \frac{\partial}{\partial u} + \frac{\partial}{\partial v}\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} \\ &= -c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) \\ &= \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + 2\frac{\partial^2}{\partial u \partial v}\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2} &= \left(-c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}\right)\left(-c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}\right) \\ &= c^2\frac{\partial^2}{\partial u^2} + c^2\frac{\partial^2}{\partial v^2} - 2c^2\frac{\partial^2}{\partial u \partial v}\end{aligned} \tag{13}$$

$$\tag{14}$$

So we can write:

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial t^2} &= c^2\frac{\partial^2\psi}{\partial x^2} && (15)\\
\left(c^2\frac{\partial^2}{\partial u^2}+c^2\frac{\partial^2}{\partial v^2}-2c^2\frac{\partial^2}{\partial u\partial v}\right)\psi &= c^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}+2\frac{\partial^2}{\partial u\partial v}\right)\psi && (16)\\
-2c^2\frac{\partial^2}{\partial u\partial v}\psi &= 2c^2\frac{\partial^2}{\partial u\partial v}\psi && (17)\\
\Rightarrow \frac{\partial^2\psi}{\partial u\partial v} &= 0 && (18)
\end{aligned}
$$

7. First, do the differentiation:

$$
\begin{aligned}
\frac{\partial\psi}{\partial t} &= i\omega\psi && (19)\\
\frac{\partial^2\psi}{\partial t^2} &= -\omega^2\psi && (20)
\end{aligned}
$$

Then substitute in:

$$
\begin{aligned}
-m\omega^2Ae^{i(\omega t+\phi)} &= -sAe^{i(\omega t+\phi)}-ib\omega Ae^{i(\omega t+\phi)}+F_0e^{i\omega t} && (21)\\
\Rightarrow \omega^2Ae^{i\phi} &= \frac{s}{m}Ae^{i\phi}+2i\gamma\omega Ae^{i\phi}-\frac{F_0}{m} && (22)
\end{aligned}
$$

where we have defined $\gamma = b/2m$. The second line follows from the first line by dividing through both sides with $me^{i\omega t}$. If we collect terms in $Ae^{i\phi}$ on the left-hand side, and note that $\omega_0 = \sqrt{s/m}$ then we have:

$$
\begin{aligned}
Ae^{i\phi}\left(\omega_0^2-\omega^2+2i\gamma\omega\right) &= \frac{F_0}{m} && (23)\\
\Rightarrow Ae^{i\phi} &= \frac{F_0}{m}\frac{1}{(\omega_0^2-\omega^2)+2i\gamma\omega} && (24)
\end{aligned}
$$

as required. The standard formulae to find the amplitude and phase follow from this expression.
The impedance is defined as $F_0/\dot{\psi}$ where we are interested in the complex amplitude of $\dot{\psi}$. We write:

$$
\begin{aligned}
Z(\omega) &= \frac{F_0}{i\omega Ae^{i\phi}} && (25)\\
&= \frac{F_0}{\frac{F_0}{m}\frac{i\omega}{(\omega_0^2-\omega^2)+2i\gamma\omega}} && (26)\\
&= \frac{m\left(\omega_0^2-\omega^2+2i\gamma\omega\right)}{i\omega} && (27)\\
&= 2m\gamma+im\left(\frac{\omega^2}{\omega}-\frac{\omega_0^2}{\omega}\right) && (28)\\
&= b+i(m\omega-s/\omega) && (29)
\end{aligned}
$$

as required; we've used $\omega_0^2 = s/m$ in the last line.

The first five questions are designed to give you some practice with ideas at the heart of the course, and are of a level with the problems sheets. The last two questions are more challenging, but will give deeper insight into the maths of the course. If you don't have time to do all the questions in the class, you should try to find time to do them before looking at the answers (which will be posted on Moodle).

1. A 1.3m steel wire with mass 0.01 kg, cross-sectional area 1mm^2 and Young's modulus 2×10^{11}Pa is kept under a tension of 800N.

   (a) What is the speed of transverse waves on the wire ? What impedance does the wire present to these waves ?

   (b) What speed and impedance do longitudinal waves have on the wire ?

2. (a) If the *intensity* of a sound increases by a factor of three (i.e. is three times larger), what will be the change in the sound level ? Give your answer in dB.

   (b) I have access to a large number of rather weak loudspeakers (which act as sources) with power 10^{-6}W/m^2. How many sources do I need to arrange in a room to provide a sound level 10dB higher than the level of one source ?

3. What is the power per unit area of:

   (a) A jet engine with sound level 130 dB ?

   (b) A room filled with conversation with sound level 50 dB ?

   How much more intense is the jet engine than the room filled with conversation ?

4. (a) Show that the frequency shift caused by a source moving towards a stationary observer can be reproduced by the observer moving towards the source.

   (b) Using the binomial theorem, derive a condition for which the two speeds will be the same.

5. The phase velocity of a surface wave in a liquid of depth much greater than λ is given by

$$v_p = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\rho\lambda}} \quad (1)$$

   where g is acceleration due to gravity, λ is wavelength, ρ is density, and γ is surface tension. Compute the group velocity of a pulse in the long-wavelength limit.

6. A piano string obeys the wave equation:

$$\frac{\partial^2\psi}{\partial t^2} = c^2\left[\frac{\partial^2\psi}{\partial x^2} - \alpha\frac{\partial^4\psi}{\partial x^4}\right] \quad (2)$$

   For a periodic wave on the string, $\psi(x,t) = A\exp i(kx - \omega t)$, show that:

   (a) $\omega = ck(1 + \alpha k^2)^{\frac{1}{2}}$

   (b) $\omega = ck - dk^3$ if $\alpha k^2 \ll 1$ and find an expression for d.

   (c) If a 1m piano string has $d/c = -1 \times 10^{-4}\text{m}^2$ and is fixed at both ends, estimate the percentage increase in the frequency of the tenth mode (i.e. the standing wave with $n = 10$) due to the k^3 term compared to a string which obeys $\omega = ck$.

7. Show that the relativistic Doppler shift:

$$f_{\text{obs}} = \sqrt{\frac{1 + v/c}{1 - v/c}} f_{\text{source}} \quad (3)$$

   reduces to the Newtonian Doppler shift for a moving observer to first order in v/c when $v \ll c$. [Hint: notice that $(1 - \beta^2) = (1 - \beta)(1 + \beta)$ and that this goes to one to first order in $\beta = v/c$.]

   [N.B. I know that this has different signs to the formula given in the lectures; here we define v as positive when observer and source are approaching each other, while in the lectures it was the other way around, for consistency with PHAS1246].

1. (a) Transverse waves are governed by tension and mass per unit length; we are given $T = 800$N and can find $\mu = 0.01/1.3 = 0.0077$kg/m. This gives a speed of $c = \sqrt{T/\mu} = \sqrt{800/0.0077} = 322.5$m/s. The impedance is defined as $Z_0 = \sqrt{T\mu} = 2.48$ kg/s.
   (b) Longitudinal waves depend on Young's modulus and density; we are given that $Y = 2 \times 10^{11}$Pa and find $\rho = 0.01/(1.3 \times 10^{-6}) = 7.69 \times 10^3$kg/m^3. The speed is then $c = \sqrt{B/\rho} = 5100$m/s. The impedance depends on area as well and is given by $Z_0 = A\sqrt{\rho Y} = 10^{-6}\sqrt{2 \times 10^{11} \times 7.69 \times 10^3} = 39.22$kg/s. Notice how different the speeds and impedances are for the two types of wave. Also notice that, while the longitudinal impedance carries an explicit area dependence, the transverse impedance carries an *implicit* dependence on area as it depends on mass per unit length (which will change with area).
2. (a) We write $I_2 = 3I_1$. Then the change in sound level is $10\log_{10}(I_2/I_1) = 10\log_{10}(3) = 4.77$ dB.
   (b) The sound level of one source is $\beta = 10\log_{10}(10^{-6}/10^{-12}) = 60$dB. To increase this by 10dB, I need a sound level of 70dB, which means an intensity of 10^{-5}W/m^2. This will require 10 sources (assuming that they add linearly).
3. For a given intensity I_1 (which is power per unit area), we have the definition that $\beta = 10\log_{10}(I_1/I_0)$ with $I_0 = 1 \times 10^{-12}$W/m^2. So $I_1 = 10^{-12}10^{(\beta/10)}$ W/m^2
   (a) $I = 10^{-12}10^{13} = 10$W/m^2
   (b) $I = 10^{-12}10^{5} = 10^{-7}$W/m^2

   The jet engine is 10^8 times more intense.
4. (a) A moving source gives a frequency of:
$$f' = f\left(\frac{v}{v - v_S}\right) \tag{1}$$
   while a moving observer gives a frequency of:
$$f' = f\left(\frac{v + v_O}{v}\right) \tag{2}$$
   We want these two to be the same, so:
$$\begin{aligned}
\left(\frac{v}{v - v_S}\right) &= \left(\frac{v + v_O}{v}\right) && (3)\\
v + v_O &= \left(\frac{v^2}{v - v_S}\right) && (4)\\
\Rightarrow v_O &= \left(\frac{v^2}{v - v_S}\right) - v && (5)\\
&= v\left(\frac{v}{v - v_S} - 1\right) && (6)\\
&= v\left(\frac{v}{v - v_S} - \frac{v - v_S}{v - v_S}\right) && (7)\\
&= v\left(\frac{v_S}{v - v_S}\right) = v\left(\frac{v_S/v}{1 - v_S/v}\right) && (8)
\end{aligned}$$
   Notice that the two speeds are *not* the same in general.
   (b) To find a condition where they are the same, we will approximate $(1 - v_s/v)^{-1}$ using the binomial theorem:
$$\begin{aligned}
v_O &= v \times \frac{v_S}{v} \times (1 - v_s/v)^{-1} && (9)\\
&= v_S(1 + v_S/v) && (10)
\end{aligned}$$
   The approximation relies on v_S/v being small, and to first order in that quantity $v_O = v_S$ if $v_S/v \ll 1$.
5. We note that long wavelength limit means that only the first term will survive, and that $\lambda/2\pi = 1/k$. Then we have:
$$\begin{aligned}
v_p &= \frac{\omega}{k} = \sqrt{\frac{g}{k}} && (11)\\
\Rightarrow \omega &= \sqrt{gk} && (12)\\
v_g &= \frac{\partial\omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}v_p && (13)
\end{aligned}$$

So the group velocity will be slower than the phase velocity.

6. We note that:

$$\begin{aligned} \frac{\partial^2\psi}{\partial t^2} &= -\omega^2\psi && (14)\\ \frac{\partial^2\psi}{\partial x^2} &= -k^2\psi && (15)\\ \frac{\partial^4\psi}{\partial x^4} &= k^4\psi && (16)\\ & && (17)\end{aligned}$$

(a) Substituting into the wave equation, we find:

$$\begin{aligned} -\omega^2\psi &= -c^2k^2\psi - c^2\alpha k^4\psi && (18)\\ \Rightarrow \omega^2 &= c^2k^2 + c^2\alpha k^4 && (19)\\ \omega &= ck(1+\alpha k^2)^{1/2} && (20)\end{aligned}$$

(b) If we have $\alpha k^2 \ll 1$ then we can approximate $(1+\alpha k^2)^{1/2} \simeq (1+\frac{1}{2}\alpha k^2)$. We can write:

$$\omega = ck(1+\frac{1}{2}\alpha k^2) = ck - dk^3 \quad (21)$$

if we define $d = -\frac{1}{2}c\alpha$.

(c) The difference in frequency will be due to the k^3 term, so that the percentage difference can be written:

$$100 \times \frac{-dk^3}{ck} = -100 \times \frac{d}{c}k^2 \quad (22)$$

Now the wavelength of the different modes for a fixed string are given by $\lambda_n = 2L/n$, and the wavenumber for $n = 10$ will be $k_{10} = 2\pi/\lambda_{10} = 10\pi/L = 31.4\mathrm{m}^{-1}$. Then the percentage increase is $100 \times 10^{-4}(31.4^2) \simeq 10\%$

7. The main trick here is to note that we can multiply top and bottom by $\sqrt{1+v/c}$. This gives:

$$\begin{aligned} f_{obs} &= f_{source}\sqrt{\frac{1+v/c}{1-v/c}\frac{1+v/c}{1+v/c}} && (23)\\ &= \frac{1+v/c}{\sqrt{(1-v/c)(1+v/c)}} && (24)\\ &= \frac{1+v/c}{\sqrt{1-(v/c)^2}} && (25)\end{aligned}$$

Then we note that, to first order in v/c, the denominator is 1, which recovers the Newtonian Doppler shift for a moving observer.

You should attempt all problems. Worked answers will be posted approximately one week after setting on Moodle.

1. (a) An electrical circuit containing an inductor (L), a resistor (R) and a capacitor (C) with a time-varying applied voltage acts as a damped, driven harmonic oscillator, with the equation:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + q/C = V_0 \cos \omega t$$

   (i) Show that the natural frequency, $\omega_0 = 1/\sqrt{LC}$ [1]

   (ii) What role does the resistor R play in the circuit, compared to an oscillator ? [1]

   (iii) The amplitude for a driven, damped harmonic oscillator is given by:

$$A = \frac{V_0}{L}\left[\left(\omega_0^2 - \omega^2\right)^2 + \frac{R^2\omega^2}{4L^2}\right]^{-\frac{1}{2}} \quad (1)$$

   For a circuit with $L = 22mH$, $C = 2.2nF$, $R = 500\Omega$ and $V_0 = 3V$, find the natural frequency, and the amplitude if the driving frequency is 1 kHz. Is this mass (impedance) dominated or stiffness (capacitance) dominated ? [4]

   (b) For a lightly damped harmonic oscillator with mass $m = 0.010$ kg and spring constant $s = 36$ N/m and general displacement $\psi(t) = A\exp(-\gamma t)\cos(\omega_f t + \phi)$ with $\gamma = b/2m$:

   (i) What value of damping constant b would make the amplitude drop from A to A/e in 1s ? [2]

   (ii) What value of b would produce critical damping ? [2]

2. (a) Consider adding together two simple harmonic oscillations at the same frequency $A_1 e^{i(\omega t+\phi_1)}$ and $A_2 e^{i(\omega t+\phi_2)}$ to give a single oscillation $Ae^{i(\omega t+\phi)}$. Use complex exponentials to show that: [3]

$$\begin{aligned} A^2 &= A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2 - \phi_1) \\ \phi &= \tan^{-1}\frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2} \end{aligned}$$

   (b) Draw phasor diagrams for and calculate the sums of the following pairs of oscillations:

   (i) $A_1 = 2, \phi_1 = 0, A_2 = 2, \phi_2 = \pi/3$ [2]

   (ii) $A_1 = 3, \phi_1 = 5\pi/4, A_2 = 2, \phi_2 = \pi/3$ [2]

   (c) A harmonic system vibrates with the following sum of two oscillations:

$$7.5\cos(6.28t + 27°) - 7.5\sin(6.20t - 120°)$$

   where time is measured in seconds. Find the frequency of the net motion, and the time interval separating successive beats. [3]

3. Two oscillators both with stiffness s and mass m are joined by a spring with stiffness K. At $t = 0$ the first oscillator (displacement ψ_1) is displaced by $\sqrt{2}A_0$ to the right (i.e. the positive ψ_1 direction) while the second oscillator (displacement ψ_2) is held fixed, and then both are released.

   (a) Show that the resulting motion can be written:

$$\begin{aligned} \psi_1 &= \frac{A_0}{\sqrt{2}}\left(\cos(\omega_a t) + \cos(\omega_b t)\right) \\ \psi_2 &= \frac{A_0}{\sqrt{2}}\left(\cos(\omega_a t) - \cos(\omega_b t)\right) \end{aligned}$$

   and give expressions for ω_a and ω_b. [Hint: work in terms of q_a and q_b first and calculate what $q_a(0)$ and $q_b(0)$ must be; as the initial velocities are zero, this will allow you to find $A_a, A_b, \phi_a \& \phi_b$; then transform to ψ_1 and ψ_2.] [6]

   (b) If $s = 81$ N/m and $K = 20$ N/m and the masses are 10 kg, show that after $t = 4.96$s the amplitude of ψ_1 will be zero. What will the amplitude of ψ_2 be ? What type of motion do the oscillators undergo ? [Hint: rewrite the solutions you found in the first part as a product of trigonometrical functions.] [4]

1. (a) i. The natural frequency for a harmonic oscillator is given by the equation:

$$m\frac{d^2\psi}{dt^2} = -s\psi \tag{1}$$

which yields $\omega_0 = \sqrt{s/m}$ In the electrical circuit, $L \equiv m$ and $\frac{1}{C} \equiv s$. The natural frequency is then $\omega_0 = 1/\sqrt{LC}$.

ii. The resistor is a damping term.

iii. The natural frequency is $\omega_0 = 1/(22\times10^{-3}\times2.2\times10^{-9})^{0.5} = 1.44\times10^5 s^{-1}$. At a driving frequency of 1 kHz, $\omega = 6.28\times10^3 s^{-1}$, $(\omega_0^2-\omega^2)^2 = (2.06\times10^{10})^2 = 4.25\times10^{20}s^{-4}$and $A = 1363/(4.25\times10^{20}+ 2.5\times10^5\times3.94\times10^7/(4*4.84\times10^{-4}))^{0.5} = 136.36/(4.25\times10^{20}+5.09\times10^{17})^{0.5} = 6.61\times10^{-9}C$. This is in the stiffness-controlled regime.

(b) i. We want $\exp(-\gamma) = 1/e$ (setting $t = 1s$), so we need $\gamma = 1 \Rightarrow b/2m = 1$. Thus $b = 2m = 0.020$ kg/s.

ii. Critical damping occurs when $\gamma = \omega_0$. Then we have $b/2m = \sqrt{s/m} \Rightarrow b = 2\sqrt{sm}$. Thus $b = 2\sqrt{0.01\times36} = 1.20$ kg/s.

2. (a) Note that, for two complex numbers z_1 and z_2,

$$\begin{aligned} |z_1+z_2|^2 &= (z_1+z_2)(z_1+z_2)^\star = (z_1+z_2)(z_1^\star+z_2^\star) && (2)\\ &= |z_1|^2+|z_2|^2+(z_1z_2^\star+z_1^\star z_2) && (3)\\ &= |z_1|^2+|z_2|^2+2\mathrm{Re}(z_1z_2^\star) && (4)\end{aligned}$$

Now we have $z_1 = A_1e^{i(\omega t+\phi_1)}$ and $z_2 = A_2e^{i(\omega t+\phi_2)}$ with A_1 and A_2 real. This gives:

$$\begin{aligned} |z_1+z_2|^2 &= |z_1|^2+|z_2|^2+2\Re(z_1z_2^\star) && (5)\\ &= A_1^2+A_2^2+2\mathrm{Re}(A_1e^{i(\omega t+\phi_1)}A_2e^{-i(\omega t+\phi_2)}) && (6)\\ &= A_1^2+A_2^2+2\mathrm{Re}(A_1A_2e^{i(\phi_1-\phi_2)}) && (7)\\ &= A_1^2+A_2^2+2A_1A_2\cos(\phi_2-\phi_1) && (8)\end{aligned}$$

while we can find the phase from:

$$\begin{aligned} \arg(z_1+z_2) &= \tan^{-1}\left[\frac{\mathrm{Im}(z_1+z_2)}{\mathrm{Re}(z_1+z_2)}\right] && (9)\\ \Rightarrow\theta &= \tan^{-1}\left[\frac{A_1\sin\phi_1+A_2\sin\phi_2}{A_1\cos\phi_1+A_2\cos\phi_2}\right] && (10)\end{aligned}$$

(b) The diagrams are shown in Fig. 1. One mark for each diagram and one each for phase and amplitude.

i. We write:

$$\begin{aligned} A^2 &= 4+4+8\cos(\pi/3) = 12.000 && (11)\\ \Rightarrow A &= \sqrt{12} = 3.464 && (12)\\ \tan\phi &= \frac{2\sin0+2\sin\pi/3}{2\cos0+2\cos\pi/3} = \frac{\sqrt{3}}{1.5} = 0.577 && (13)\\ \Rightarrow\phi &= \pi/6 && (14)\end{aligned}$$

(though note that we have a standard formula in the notes which gives $\phi = \phi_2/2$ and $A = 2A\cos(\phi_2/2)$ for equal amplitudes which is also an acceptable route, and should give the same answer).

ii.

$$\begin{aligned} A^2 &= 9+4+12\cos(11\pi/12) = 1.409 && (15)\\ \Rightarrow A &= \sqrt{1.409} = 1.187 && (16)\\ \tan\phi &= \frac{3\sin5\pi/4+2\sin\pi/3}{3\cos5\pi/4+2\cos\pi/3} = \frac{\sqrt{3}}{1.5} = 0.347 && (17)\\ \Rightarrow\phi &= \pi+0.334 = 1.106\pi && (18)\end{aligned}$$

where we must add π to the phase because of the relative signs of the phasors and the periodicity of tan. A quick check with the phasor diagram shows that this is needed.

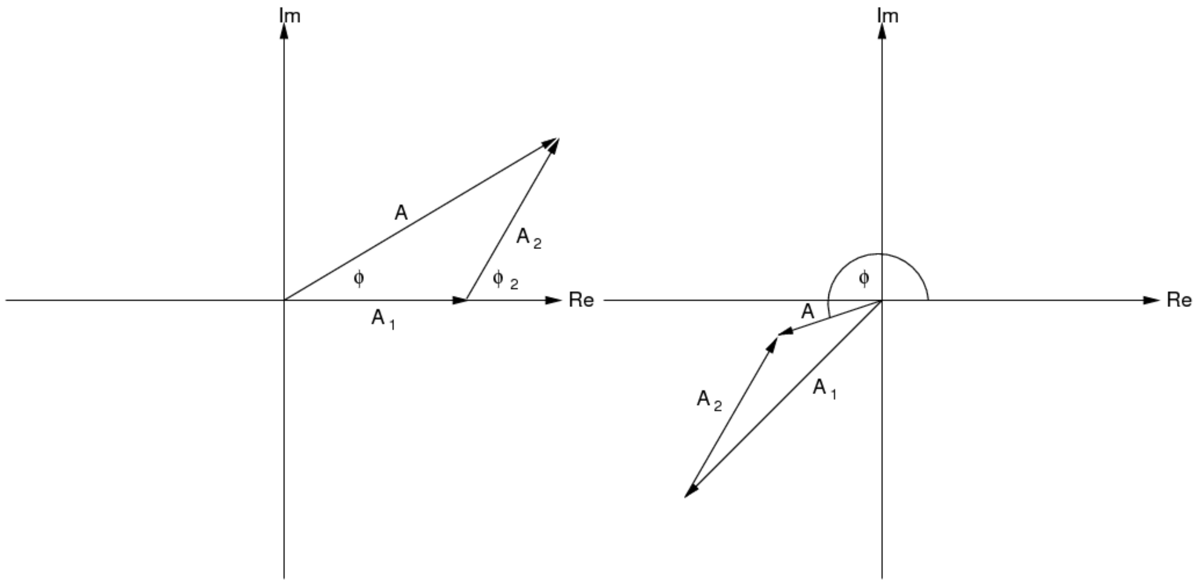

Figure 1: Phasor diagrams for question 2(b)

(c) We must convert degrees to radians; I find it easier to change the sin to a cos using a phase shift of $\pi/2$ (so $\sin(x) = \cos(x - \pi/2)$. Then we can combine the sum of two cosines into a product. So:

$$\begin{aligned} 27^\circ &= 0.15\pi && (19)\\ 121^\circ &= 2\pi/3 && (20)\\ 7.5\left(\cos(6.28t + 0.15\pi) - \sin(6.20t - 2\pi/3)\right) &= 7.5\left(\cos(6.28t + 0.15\pi) - \cos(6.20t - 7\pi/6)\right) && (21)\\ &= 15\sin(6.24t - 0.508\pi)\sin(0.04t + 0.658\pi) && (22) \end{aligned}$$

where we have used a standard rule for cosines. So the frequency of the net motion will be $6.24/2\pi$ s^{-1} or 0.99 s^{-1} and the time between successive beats will be 78.54s (we take $2\pi/\Delta\omega$, with $\Delta\omega = 0.08\text{s}^{-1}$). Note that the time between beats is given by $\omega_1 - \omega_2$ not $(\omega_1 - \omega_2)/2$ - we are interested in the amplitude modulation, not the wave motion.

3. (a) i. We know that we can solve the coupled harmonic oscillators with the following quantities defined in lectures:

$$q_a = (\psi_1 + \psi_2)\sqrt{\frac{m}{2}} = A_a\cos(\omega_a t + \phi_a) \quad (23)$$

$$q_b = (\psi_1 - \psi_2)\sqrt{\frac{m}{2}} = A_b\cos(\omega_b t + \phi_b) \quad (24)$$

where $\omega_a = \sqrt{s/m}$ and $\omega_b = \sqrt{(2K+s)/m}$. If we set $t = 0$ and the initial velocities to zero, then we have:

$$\begin{aligned} q_a &= A_a\cos(\phi_a) = \sqrt{m}A_0 && (25)\\ q_b &= A_b\cos(\phi_b) = \sqrt{m}A_0 && (26)\\ \dot{q}_a &= A_a\sin(\phi_a) = 0 && (27)\\ \dot{q}_b &= A_b\sin(\phi_b) = 0 && (28) \end{aligned}$$

That is enough information to tell us that $\phi_a = \phi_b = 0$ and that $A_a = A_b = \sqrt{m}A_0$ as $\psi_1(0) = \sqrt{2}A_0$. We notice that *both* modes are equally excited. Then we can recover ψ_1 and ψ_2 from the definitions above ($\psi_1 = (q_a + q_b)/\sqrt{2m}$ and $\psi_2 = (q_a - q_b)/\sqrt{2m}$) and write:

$$\psi_1 = \frac{A_0}{\sqrt{2}}\left(\cos(\omega_a t) + \cos(\omega_b t)\right) \quad (29)$$

$$\psi_2 = \frac{A_0}{\sqrt{2}}\left(\cos(\omega_a t) - \cos(\omega_b t)\right) \quad (30)$$

as required.

ii. We start by finding $\omega_a = \sqrt{81/10} = 2.846\text{s}^{-1}$ and $\omega_b = \sqrt{121/10} = 3.479\text{s}^{-1}$. Then we notice that we can rewrite $\cos(\omega_a t) + \cos(\omega_b t) = 2\cos(\omega t)\cos(\Delta\omega t)$, with $\omega = (\omega_a + \omega_b)/2$ and $\Delta\omega = (\omega_a - \omega_b)/2 = 0.317\text{s}^{-1}$. Then the amplitude of ψ_1 will be zero when $\Delta\omega t = \pi/2$ which gives $t = \pi/(2 \times 0.317) = 4.96\text{s}$. The amplitude of ψ_2 will be at a maximum with value $\sqrt{2}A_0$ (because we can write $\cos(\omega_a t) - \cos(\omega_b t) = 2\sin(\omega t)\sin(\Delta\omega t)$). The type of motion is *beats*.

You should attempt all problems. Worked answers will be posted approximately one week after setting on Moodle.

1. (a) Two sinusoidal waves travel in the same direction along a string and interfere to produce a resultant wave given by:

$$y(x,t) = (3.0mm)\sin(20x - 4.0t + 0.82)$$

with x in meters and t in seconds. The two waves are identical, except for phase; one of the waves has zero phase. What is:

   (i) The wavelength of the two waves ? [2]

   (ii) The phase difference between them ? [2]

   (iii) The amplitude of the waves ? [2]

   (b) A disturbance $y_1(x,t) = A\cos(kx - \omega t)$ is superposed on another disturbance $y_2(x,t) = A\cos(k'x - \omega' t)$. Derive an expression for the resultant, and sketch it. [4]

2. (a) A wave of frequency 500 Hz has a velocity of 350 m/s.

   (i) How far apart are two points that differ in phase by $\pi/3$? [2]

   (ii) What is the phase difference between two displacements at a certain point at times 1 ms apart? [3]

   (b) Write an expression describing a sinusoidal transverse wave travelling on a string in the -y direction with a wavelength of 60cm, a period of 0.20s and an amplitude of 3mm. [3]

   (c) Calculate the tension and mass per unit length of a string which has a characteristic impedance of $Z_0 = 3$ kg/s and phase velocity for waves of $c = 30$ m/s. [2]

3. (a) Two strings with mass 1kg/m and 2kg/m are joined together. If the two are put under a tension of 20 N/m, and a wave pulse of amplitude 1cm is sent down the lighter string towards the join, what will be the amplitude on both strings *after* the wave pulse reaches the join ? [4]

   (b) If a co-axial extension cable with characteristic impedance 120Ω is joined to an aerial cable with characteristic impedance 75Ω, what amplitude of signal will be received at the end of the extension cable if a signal of 100 μV is received at the aerial ? [Hint: you can treat the voltage as the amplitude of a wave and the impedances in just the same way as the impedance on a string] [4]

   (c) If I have the two transmission lines in the previous question, which way around should I connect them to maximise the transmitted amplitude ? [2]

1. (a) We are going to use the formula for combining oscillations, $y = A\cos(\theta+\phi) - A\cos(\theta) = 2A\sin(\phi/2)\sin(\theta + \phi/2)$

   i. The wavelength is $2\pi/k = 2\pi/20 = 0.31$m

   ii. The phase difference between them is $2 \times 0.82 = 1.64$ radians

   iii. We have that $2A\sin(0.82) = 0.003$, so $A = 0.0021$m

   (b) We have $\cos A + \cos B = 2\cos(A+B)/2\cos(A-B)/2$, so the resultant will be:

$$y_R(x,t) = 2A\cos(k_a x - \omega_a t)\cos(\Delta k x - \Delta\omega t)$$

   where $k_a = (k+k')/2$, $\Delta k = (k-k')/2$ and similarly for ω. The resultant will look like Fig. 1.

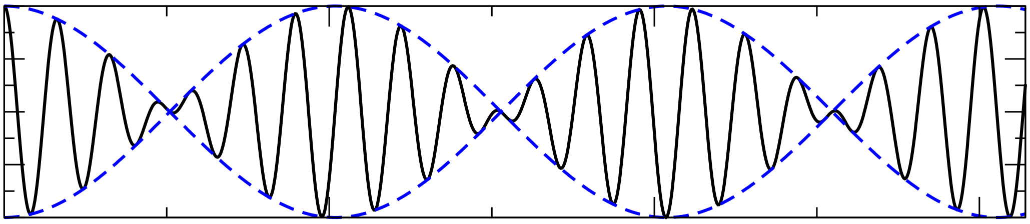

Figure 1: Resultant wave, with the horizontal axis being x or t and the vertical axis y.

2. (a) First, note that the wavelength is $\lambda = c/\nu = 350/500 = 0.7$m.

   i. We have $\pi/3 = 2\pi\Delta x/\lambda$ so $\Delta x = \lambda/6 = 0.1167$m.

   ii. We have $\Delta\phi = 2\pi\nu\Delta t = \pi$ in radians.

   (b) $f(x) = 0.003\cos(10.47y + 31.42t)$

   (c) We have from lectures that $Z_0 = \sqrt{T\mu}$ and $c = \sqrt{T/\mu}$ so $T = Z_0 c = 90$ N and $\mu = Z_0/c = 0.1$kg/m.

3. (a) The key quantities are the characteristic impedances. We have from lectures that $Z_0 = \sqrt{T\mu}$ which will be 4.472 kg/s and 6.325 kg/s for the two strings. Then the reflection coefficient will be $(4.472 - 6.325)/(4.472 + 6.325) = -0.172$ (and a consistency check tells us that it's OK to have a negative sign - this indicates the phase reversal we'll get going from lower impedance to higher). The transmission coefficient is $2 \times 4.472/(4.472 + 6.325) = 0.828$. So the reflected and transmitted pulses will have amplitudes of 0.172cm and 0.828cm respectively.

   (b) We want the transmitted amplitude, which will be given by the transmission coefficient ($T = 2 \times 75/(75 + 120) = 0.769$) multiplied by the incoming amplitude so the signal will have amplitude 76.9 μV.

   (c) The transmission coefficient is given by $T = 2Z_1/(Z_1 + Z_2)$ so we want the transmission line with the *higher* impedance on the transmitted side.

You should attempt all problems. Worked answers will be posted approximately one week after setting on Moodle.

1. (a) An earthquake emits a longitudinal wave which is felt 5,000km from the epicentre 15 minutes after the event. If the average density of the earth's crust is $\rho = 2.7 \times 10^3$ kg/m^3, calculate the average Young's modulus of the crust. [3]

   (b) Two organ pipes have lengths 2m and 4m. The 2m pipe is closed at one end, while the 4m pipe is open at both ends. Find the wavelengths of the first four modes in each pipe, and say which modes, if any, will share the same frequency. [4]

   (c) The compressibility of water is $4.9{\times}10^{-10}$m^2/N.

      (i) Estimate the speed of sound in water

      (ii) Estimate the characteristic impedance per unit area for water

      (iii) When swimming underwater, why can you not hear sounds in the air at the surface ? [The impedance of air at room temperature and atmospheric pressure is around 400 kg/sm^2.] [3]

2. In this question, assume that the speed of sound in air is 343 m/s.

   (a) A shower stall, closed at the top, is 2.40 m tall. For what frequencies less than 300 Hz are there standing sound waves in the shower stall with the door is closed? [3]

   (b) If the top of the stall is open, for what frequencies less than 300 Hz are there standing sound waves in the shower stall? [3]

   (c) Draw the graphical representations of the standing waves for the first two frequencies found in Questions 2(a) and 2(b). [4]

3. (a) (i) Microwaves form standing waves in a microwave oven. Cold spots in a microwave oven are found to be 6.2 cm apart. What is the frequency of the microwaves ? [You may assume that the speed of light is given by $c = 3 \times 10^8$ m/s] [2]

      (ii) What are the three longest wavelengths for standing waves on a 260-cm long string that is fixed at both ends? [3]

      (iii) If the frequency of the second-longest wavelength is 60 Hz, what is the frequency of the third-longest wavelength? [2]

   (b) A horizontal string, tied to a sinusoidal oscillator at a point P and running over a support at Q, is stretched by a block of mass m. The separation L between P and Q is 1.2 m, the linear density of the string is 1.6 g m^{-1}, and the frequency f of the oscillator is fixed at 120 Hz. P and Q can be considered to be nodes. What mass m allows the oscillator to set up the fourth harmonic on the string? [3]

1. (a) We know that the speed of longitudinal waves in a solid is given by $c = \sqrt{Y/\rho}$. This means that:

$$c^2 = Y/\rho \quad (1)$$
$$Y = c^2\rho \quad (2)$$

The speed of the wave is $5 \times 10^6/(15 \times 60) = 5,555.6$m/s. This gives us $Y = 2.7 \times 10^3 \times (5555.6)^2 = 8.3 \times 10^{10}$Pa.

(b) The open pipe will have wavelengths given by $\lambda_n = 2L/n$ for $n = 1, 2, 3, 4$, so the first four wavelengths will be 8m, 4m, 2.67m, 2m .

The closed pipe will have wavelengths $\lambda_n = 4L/n$ for $n = 1, 3, 5, 7$. The first four wavelengths will be 8m, 2.67m, 1.6m, 1.14m. The first modes on both will share the same frequency, as will the second mode on the closed pipe and the third mode on the open pipe. Note that we assume the same speed of wave in both cases.

(c) i. We know that the density of water is around 1,000 kg/m^3, so the speed of sound will be $\sqrt{1/(1000 \times 4.9 \times 10^{-10})} = 1.43 \times 10^3$m/s.

ii. Per unit area, we have $Z_0 = \sqrt{1000/4.9 \times 10^{-10})} = 1.43 \times 10^6$kg/m^{2}s.

iii. The mismatch in impedance will make the transmission coefficient very small (of the order of 10^{-4}).

2. (a) The shower stall is closed at both ends, so

$$f_1 = \frac{c}{2L} = \frac{343}{2(2.40)} = 71.5\text{Hz} \quad (3)$$

The possible standing wave frequencies, in Hz, are integer multiples of this fundamental: 71.5, 143, 214.5, and 286.

(b) Now we have a stall closed at one end and open at the other, so

$$f_1 = \frac{c}{4L} = \frac{343}{4(2.40)} = 35.7\text{Hz} \quad (4)$$

The possible standing wave frequencies, in Hz, are odd-integer multiples of this fundamental: 35.7, 107.1, 178.5, and 249.9.

(c) Sketches in Fig. 1.

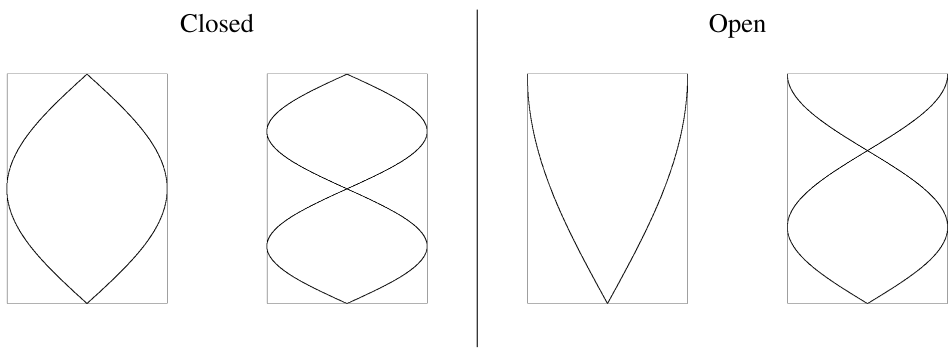

Figure 1: Shower stall modes: closed (left) and open (right)

3. (a) i. Between the nodes, or cold spots, we have

$$\frac{\lambda}{2} = 6.2cm \Rightarrow \lambda = 12.4cm \quad (5)$$

Then, since $c = f\lambda$, $f = \frac{c}{\lambda} = \frac{3\times10^8}{0.124} = 2.42 \times 10^9$ Hz

ii. $\lambda_1 = 2L = 2 \times 2.6 = 5.2$m, $\lambda_2 = \frac{2L}{2} = \frac{5.2}{2} = 2.6$m and $\lambda_3 = \frac{2L}{3} = \frac{5.2}{3} = 1.7$m.

iii. Since $f_m = mf_1$, $f_3 = 3 \times f_1 = 3 \times \frac{f_2}{2} = 3 \times 60/2 = 90$Hz.

(b) By specifying the wavelength (fourth mode) and frequency (from the oscillator) we have implicitly set the velocity required; since we can vary the tension by varying the mass, we will be able to choose the velocity of the waves on the string, and hence which mode is excited by the oscillator . For a standing wave on a string

with nodes at both ends, we can define $f_n = n(v/2l)$ for $n = 1, 2, 3, \ldots$ (which comes from the definition of the wavelength, $\lambda_n = 2L/n$).
This gives an expression for the speed of the wave in a particular mode as:

$$v = 2lf_n/n \tag{6}$$

Now for this system the tension τ will be provided by the weight of the mass m; we need to work out what tension (and hence what mass) will give a wave speed equivalent to the fourth mode. We can write :

$$v = \sqrt{\tau/\mu} = \sqrt{mg/\mu} \tag{7}$$

so, rearranging, we find:

$$m = \frac{v^2\mu}{g} \tag{8}$$

Now we substitute in for v and then the physical parameters:

$$m = \frac{4l^2f_4^2\mu}{n^2g} = \frac{4 \times 1.2^2 \times 120^2 \times 0.0016}{4^2 \times 9.8} = 0.846\text{kg} \tag{9}$$

You should attempt all problems. Worked answers will be posted approximately one week after setting on Moodle.

1. (a) If you are given two sound intensities, I_1 and I_2, how is the difference in the sound levels β_1 and β_2 related to the intensities ? [4]

   (b) By what factor has the intensity increased if the sound level has risen by 20 dB? [3]

   (c) In 1976, the Who set a record for the loudest concert: the sound level 46m in front of the speaker system was $\beta_2 = 120$ dB. What is the ratio of the intensity I_2 of the band at that spot to the intensity I_1 of a jackhammer operating at sound level $\beta_1 = 92$ dB (assume that the measurements are made at the same distance) ? [3]

2. (a) A police siren has a frequency of 550 Hz as the police car approaches you, 450 Hz after it has passed you and is receding.

   (i) How fast are the police traveling? [3]

   (ii) What is the siren frequency at rest? [2]

   Assume that the speed of sound is 343 m/s.

   (b) Show that, for a source emitting sound at frequency f and moving with speed v_S towards an observer, the frequency perceived will be: [5]

$$f' = f\left(\frac{v}{v - v_S}\right)$$

3. (a) Show that, for a medium with dispersion relation $\omega = ak^r$ the group velocity is $v_g = rv_p$ for all frequencies. [3]

   (b) An ionized gas or plasma is a dispersive medium for electromagnetic waves. Given that the dispersion relation is $\omega^2 = \omega_p^2 + c^2k^2$ where ω_p is the constant plasma frequency, show that $c^2 = v_p v_g$. [3]

   (c) The dispersion relation for waves in water of depth x depends on the acceleration due to gravity, g, the surface tension, Γ and the density of the water, ρ. The dispersion relation is:

$$\omega^2 = \left(g + \frac{\Gamma k^2}{\rho}\right) k \left(\frac{e^{kx} - e^{-kx}}{e^{kx} + e^{-kx}}\right)$$

   Show that the phase velocity

   (i) in the limit of very deep water is given by: [2]

$$v_p = \sqrt{\frac{1}{k}\left(g + \frac{\Gamma k^2}{\rho}\right)}$$

   (ii) in the limit of shallow water is given by: [2]

$$v_p = \sqrt{x\left(g + \frac{\Gamma k^2}{\rho}\right)}$$

1. (a) We know that a sound level is given by $\beta_1 = 10\log(I_1/I_0)$. Then we can write:

$$\begin{aligned}
\beta_2 &= 10\log_{10}(I_2/I_0) \\
\beta_1 &= 10\log_{10}(I_1/I_0) \\
\beta_2 - \beta_1 &= 10\log_{10}(I_2/I_0) - 10\log_{10}(I_1/I_0) \\
&= 10\left[\log_{10}(I_2) - \log_{10}(I_0) - (\log_{10}(I_1) - \log_{10}(I_0))\right] \\
&= 10\left[\log_{10}(I_2) - \log_{10}(I_0) - \log_{10}(I_1) + \log_{10}(I_0)\right] \\
&= 10\left[\log_{10}(I_2) - \log_{10}(I_1)\right] = 10\log_{10}(I_2/I_1) \\
\Rightarrow I_2 &= I_1 10^{(\beta_2-\beta_1)/10}
\end{aligned}$$

(b) We have that $\beta_2 - \beta_1 = 20$. Therefore $I_2 = I_1 \times 10^2$ so the intensity is 100 times greater.

(c) We have a similar calculation. $I_2/I_1 = 10^{(120-92)/10} = 631$.

2. (a) We will use the formula for moving source.

i. We have that:

$$\begin{aligned}
550 &= f\left(\frac{v}{v - v_S}\right) \\
450 &= f\left(\frac{v}{v + v_S}\right) \\
\frac{550}{450} &= \left(\frac{v + v_S}{v - v_S}\right) \\
\frac{11}{9}(v - v_S) &= v + v_S \Rightarrow \frac{20}{9}v_S = \frac{2}{9}v
\end{aligned}$$

So the speed of the police car is $0.1 \times 343 = 34.3$m/s.

ii. We get the frequency at rest using the formula: $550 = 343f/308.7 \Rightarrow f = 495$Hz.

(b) If the source moves with speed v_S then the spacing between waves will change (decreasing in the direction the source moves and increasing in the opposite direction). This will lead to a change in the measured wavelength, and as the velocity of the waves is unchanged, the frequency will change. In one period, the source will move $v_S T$ which is also the change in wavelength, $\Delta\lambda = v_s T$; if the source moves *towards* the observer the wavelength *decreases*. Then we can find the changed frequency:

$$\begin{aligned}
\lambda' &= \lambda - \Delta\lambda = \lambda - v_s T \\
&= \lambda - \frac{v_s}{f} = \frac{v}{f} - \frac{v_s}{f} \\
f' &= \frac{v}{\lambda'} = \frac{v}{\frac{1}{f}(v - v_S)} \\
&= f\left(\frac{v}{v - v_S}\right) = f\left(\frac{1}{1 - v_s/v}\right)
\end{aligned}$$

3. (a) We have:

$$v_p = \frac{\omega}{k} = \frac{ak^r}{k} = ak^{r-1} \tag{1}$$

$$v_g = \frac{\partial\omega}{\partial k} = ark^{r-1} \tag{2}$$

$$rv_p = rak^{r-1} = v_g \tag{3}$$

as required.

(b) Taking the differential with respect to k:

$$\omega^2 = \omega_p^2 + c^2k^2 \Rightarrow 2\omega\frac{d\omega}{dk} = 2c^2k \tag{4}$$

Dividing both sides by 2k gives:

$$\frac{\omega}{k}\frac{d\omega}{dk} = c^2 \tag{5}$$

But $\omega/k = v_p$ and $d\omega/dk = v_g$ so, as required,

$$v_p v_g = c^2. \tag{6}$$

(c) The limiting behaviour of $\tanh kx$ is key here (the ratio of exponentials on the right hand side is tanh).

i. As $x \to \infty, \tanh kx \to 1$ (think about what happens to the individual exponentials if you can't see this immediately). So we have:

$$\omega^2 = \left(g + \frac{\Gamma k^2}{\rho}\right) k \quad (7)$$

$$\frac{\omega^2}{k^2} = v_p^2 = \frac{1}{k}\left(g + \frac{\Gamma k^2}{\rho}\right) \quad (8)$$

$$\Rightarrow v_p = \sqrt{\frac{1}{k}\left(g + \frac{\Gamma k^2}{\rho}\right)} \quad (9)$$

ii. As $x \to 0, \tanh kx \to kx$ (expand out the exponentials to first order in kx to see this). So:

$$\omega^2 = \left(g + \frac{\Gamma k^2}{\rho}\right) k^2 x \quad (10)$$

$$\frac{\omega^2}{k^2} = v_p^2 = \left(g + \frac{\Gamma k^2}{\rho}\right) x \quad (11)$$

$$\Rightarrow v_p = \sqrt{x\left(g + \frac{\Gamma k^2}{\rho}\right)} \quad (12)$$