UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sc.

Applied Mathematical Finance

COURSE CODE	:	**MATHGM21**
DATE	:	**19-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-N0136-3-40

TURN OVER

There are six questions in this examination.
Full marks can be obtained from complete answers to four questions.
The use of calculators is permitted in this examination.

You may assume throughout this examination that dX is an increment in a standard Brownian motion $X(t)$:

$$\begin{aligned} \mathbb{E}[dX] &= 0 \\ \mathbb{E}\left[dX^2\right] &= dt. \end{aligned}$$

You may use the result

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

without proof.

Leibniz Rule: Given a function of two variables $F(s,x)$ which is continuous, then

$$\frac{\partial}{\partial x} \int_a^x F(s,x)\, ds = F(x,x) + \int_a^x \frac{\partial F(s,x)}{\partial x} ds.$$

1. (a) Use Itô's lemma to deduce the following formula for stochastic differential equations and stochastic integrals

$$\int_0^t \frac{\partial F}{\partial X} dX(\tau) = F(X(t), t) - F(X(0), 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2} \right) d\tau$$

for a function $F(X(\tau), \tau)$ where $dX(\tau)$ is an increment of a Brownian motion. Hence if $X(0) = 0$, evaluate

$$\int_0^t X^3 dX(\tau).$$

(b) Consider the Itô integral of the form

$$\int_0^T f(t, X(t))\, dX(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i).$$

The interval $[0, T]$ is divided into N partitions with end points

$$t_0 = 0 < t_1 < t_2 < \ \ldots\ldots\ < t_{N-1} < t_N = T,$$

where the length of an interval $t_i - t_{i+1}$ tends to zero as $N \to \infty$.
We know from Itô's lemma that

$$2\int_0^T X(t)\ dX(t) = X(T)^2 - X(0)^2 - T.$$

Show from the definition of the Itô integral that the result can also be found by initially writing the integral

$$2\int_0^T X(t)\, dX(t) = \lim_{N\to\infty} 2\sum_{i=0}^{N-1} X_i (X_{i+1} - X_i).$$

Hint: use $2b(a-b) = a^2 - b^2 - (a-b)^2$. Full details of all working should be given.

(c) Show that $F = \arcsin(2aX(t) + \sin F_0)$ is a solution of the stochastic differential equation

$$dF = 2a^2 (\tan F)\left(\sec^2 F\right) dt + 2a(\sec F)\, dX,$$

where $F_0 = F(0)$, $X(0) = 0$ and a is a constant.

2. (a) The Vasicek model for the spot interest rate $r(t)$ is defined, by the process

$$dr = (\eta - \gamma r)\, dt + \sigma dX$$

where γ is the reversion rate and η/γ is the mean rate.
Show that this follows a mean-reverting Ornstein-Uhlenbeck process U, which satisfies the linear stochastic differential equation

$$dU = -\gamma U dt + \sigma dX.$$

By using an integrating factor method, show that

$$U(t) = \alpha e^{-\gamma t} + \sigma \int_0^t \exp(-\gamma (t - \tau))\, dX(\tau).$$

where $U(0) = \alpha$.
Using integration by parts, further simplify the solution for $U(t)$, in terms of an integral which is purely time-dependent (and not with respect to $dX(t)$).
What is the final form for $r(t)$?

(b) Consider the diffusion process for the spot rate r which evolves according to the stochastic differential equation

$$dr = -ardt + bdX.$$

Both a and b are constants. Write down the forward Fokker-Planck equation for the transition probability density function $p(r', t')$ for this process, where a primed variable refers to a future state/time.
By solving the Fokker-Planck equation which you have obtained, obtain the **steady state** probability distribution $p_\infty(r')$, which is given by

$$p_\infty = \sqrt{\frac{a}{b^2 \pi}} \exp\left(-\frac{a}{b^2} r'^2\right).$$

3. (a) Consider two financial assets S_1 and S_2 which evolve according to the diffusion processes

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dX_1 \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dX_2 \end{aligned}$$

$\mu_1,\ \mu_2,\ \sigma_1,\ \sigma_2$ are constants and $\mathbb{E}\left[dX_1 dX_2\right] = 0$. By constructing a portfolio of the form

$$\Pi = V - \Delta_1 S_1 - \Delta_2 S_2$$

obtain the following partial differential equation for the fair price of an option $V\left(S_1,\ S_2,\ t\right)$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} = r\left(V - S_1\frac{\partial V}{\partial S_1} - S_2\frac{\partial V}{\partial S_2}\right)$$

where r is the risk-free interest rate.

(b) The Black–Scholes formula for the value of a put option $P\left(S, t\right)$ is

$$P\left(S,t\right) = E\exp(-r(T-t))N(-d_2) - SN(-d_1)$$

from this expression, find the value of this put option in the following limits:

(i) (time tends to expiry) $t \longrightarrow T$, $\sigma > 0$;

(ii) (volatility tends to zero) $\sigma \longrightarrow 0$, $t < T$;

(iii) (volatility tends to infinity) $\sigma \longrightarrow \infty$, $t < T$;

(iv) (expiry tends to infinity) $T \longrightarrow \infty$, $\sigma > 0$ and finite (this depends on the sign of $r - \frac{1}{2}\sigma^2$)

4. (a) Consider the Cauchy-Euler equation

$$aS^2\frac{d^2V}{dS^2}+bS\frac{dV}{dS}+cV=0,$$

where a , b and c are constants. Given that $(a-b)^2>4ac$, obtain the general solution

$$V(S)=AS^{\alpha_+}+BS^{\alpha_-}$$

where A and B are arbitrary constants and

$$\alpha_\pm=\frac{1}{2a}\left((a-b)\pm\sqrt{(a-b)^2-4ac}\right).$$

What are the solutions of the differential equation for the case:

(i) $(a-b)^2=4ac$

(ii) $(a-b)^2<4ac$

(b) A perpetual American call option $V(S,t)$ gives its holder the right to buy the underlying asset S for a given strike price E at any time in the future. It has no expiry date. Assume the Black-Scholes equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+(r-D)\,S\frac{dV}{dS}-rV=0 \qquad (4.1)$$

where $r>0$ is the constant interest rate, $D\geq 0$ the constant dividend yield, $\sigma>0$ is the constant volatility.

Assume that the solution of a Cauchy-Euler equation for $V(S)=S^\alpha$ as in part (a). Consider the special case when

$$r=D=\frac{1}{2}\sigma^2.$$

For this case, show that the price of a perpetual American Call option V_C with payoff $\max(S-E,\ 0)$ is

$$V_C(S)=(S^*-E)\left(\frac{S}{S^*}\right)^{\alpha_+}$$

where

$$S^*=\frac{\alpha_+}{\alpha_+-1}E=\frac{3+\sqrt{5}}{2}E \text{ and } \alpha_\pm=\frac{1\pm\sqrt{5}}{2}$$

and S^* denotes the (unknown) optimal exercise boundary (i.e. for $S\geq S^*$ the option should be exercised and for $S<S^*$ the option should be held).

5. (a) Suppose the spot interest rate r satisfies the stochastic differential equation

$$dr = u(r,t)\,dt + \omega(r,t)\,dX. \tag{5.1}$$

Derive the bond pricing equation (by hedging with a bond of different maturity) for a security $Z = Z(r\ ,t\ ;T)$

$$\frac{\partial Z}{\partial t} + \frac{1}{2}\omega^2 \frac{\partial^2 Z}{\partial r^2} - a(r,t)\frac{\partial Z}{\partial r} - rZ = 0, \tag{5.2}$$

where $a(r,t)$ is an arbitrary function of r and t.

(b) Assuming now that a is a function of t only and a bond at maturity $t = T$ has payoff of one unit, i.e.

$$Z(r\ ,T\ ;T) = 1$$

find a solution for (5.2) of the form

$$Z(r\ ,t\ ;T) = \exp(A(t) + rB(t)),$$

where $Z(r\ ,t\ ;\ T)$ is the price of a zero-coupon bond of maturity T.
Show that

$$B(t;T) = (T-t), \quad A(t\ ;T) = -\int_t^T a(\tau)(T-\tau)\ d\tau + \frac{1}{6}\omega^2 (T-t)^3.$$

(c) Hence deduce that $a(t)$ can be obtained from zero-coupon bond prices in the form

$$a(t) = \frac{\partial^2}{\partial T^2}(\log Z(r,t\ ;T)) + (T-t).$$

6. Assume that we hold a quantity of foreign currency S which receives interest at the constant foreign rate of interest r_f , and can be modelled according to the stochastic differential equation

$$\frac{dS}{S} = (\mu - r_f)\, dt + \sigma dX$$

where μ and σ are constants. The asset pays out a continuous rate of interest equal to $r_f S dt$ during the infinitessimal time interval dt.

(a) Now suppose a European option is written on this asset with the properties that at expiry the holder receives the asset and prior to expiry the option pays a continuous cash flow $C(S,t)\ dt$ during each time interval of length dt.
By setting up an instantaneously risk-free portfolio and using a no-arbitrage argument, show that the value V of the option satisfies the following equation

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - r_f)\, S \frac{\partial V}{\partial S} - rV &= -C(S,t), \\ V(S,T) &= S \end{aligned} \qquad (6.1)$$

where T is the expiry of the option.

Hint: During each time step the change in the value of the portfolio due to the continuous cash flow will be $+C(S,t)\, dt$.

(b) Suppose that $C(S,t)$ in equation (6.1) has the form $C(S,t) = f(t)\, S$.
By writing $V = \phi(t)\, S$ show that the delta of the derivative security is

$$\Delta(S,t) = \exp(-r_f (T-t)) + \int_t^T \exp(-r_f (\tau - t))\, f(\tau)\, d\tau.$$