UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sc.

Theory of Traffic Flow

COURSE CODE	:	**MATHG501**
DATE	:	**28-APR-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-N0127-3-30

TURN OVER

*All questions may be answered, but only marks obtained on the best **four** questions will count.*
*The use of an electronic calculator is **not** permitted in this examination.*

1 You may assume that for any strictly positive a_i $(i = 1, 2, \ldots, I)$ and b_j $(j = 1, 2, \ldots, J)$ such that $\Sigma_i a_i = \Sigma_j b_j$, any real $I \times J$ matrix (c_{ij}) and any real α there is an $I \times J$ matrix $\mathbf{t}^*(\alpha)$ with row and column sums a_i and b_j and such that for all i and j, $t^*_{ij} = r_i s_j \exp(-\alpha c_{ij})$ for some r_i and s_j.

Let D be the set of all $I \times J$ matrices $\mathbf{t} = (t_{ij})$ having row sums a_i $(i = 1, 2, \ldots, I)$ and column sums b_j $(j = 1, 2, \ldots, J)$ and such that $t_{ij} > 0$ for all i and j.

For $\mathbf{t}$ in D, let $F(\mathbf{t}) = \Sigma_{ij} t_{ij} \ln t_{ij} + \alpha \Sigma_{ij} c_{ij} t_{ij}$.

(a) Show that for $\mathbf{t}$ in D the matrix of second derivatives of $F(\mathbf{t})$ with respect to the IJ variables t_{ij} is positive definite.

(b) State without proof two consequences of (a) concerning stationary points of F in D.

(c) By constructing and differentiating a suitable Lagrangian function, show that $\mathbf{t}^*(\alpha)$ is a stationary point of F in D.

Use the Furness procedure to calculate $\mathbf{t}^*(0)$ when $I = 2$, $J = 3$, $a_1 = a_2 = 600$, $b_1 = 300$, $b_2 = 400$ and $b_3 = 500$.

The c_{ij} are costs of travel and the t_{ij} are numbers of journeys between origins i and destinations j in a city. State briefly the interpretation of the two terms in the function $F(\mathbf{t})$. How would you expect the relevant value of α to change over, say, 20 years during which the average income of the inhabitants of the city increased substantially?

2 Wardrop's first principle of route choice is that "the journey times on all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route."

In a network in which traffic respects the first-in first-out (FIFO) discipline, the demand for travel varies over time. Show that, in order for traffic flows in this network to satisfy Wardrop's first principle, the proportion $\mu_p(s)$ of demand that is assigned to route p at time s satisfies

$$\mu_p(s) = \frac{g_p[\tau(s)]}{\sum_{q \in P} g_q[\tau(s)]} \quad \forall p \in P,$$

where $\tau(s)$ is the time of arrival of traffic that departs at time s,

$g_p(t)$ is the outflow from route p at time t,

P is the set of routes available for that journey.

Discuss the use in the right-hand side of this expression of route outflows at time $\tau(s)$ to calculate assignment proportions at time $s < \tau(s)$.

3 Customers arrive at a queue for a certain facility according to a Poisson process with mean rate q and have mutually independent service times exponentially distributed with mean s^{-1}. If the facility is occupied when a customer arrives, then the customer goes elsewhere; otherwise the customer occupies the service facility immediately. Show that the probability of the facility being occupied at time $t \geq 0$ is given by

$$P_B(t) = [1 + \exp\{-(q+s)t\}]\left(\frac{q}{s+q}\right) + \exp\{-(q+s)t\} P_B(0).$$

Show that this probability changes less rapidly as time increases. Hence or otherwise show that the rate of change of mean occupancy of the service facility lies in the range $[-s, q]$, and identify a case in which each of the extreme values $-s$ and q is attained.

4 (a) Explain what is meant by each of a *shock wave* and a *wave* in traffic, and establish an expression for the speed at which each of these travels.

The flow of a stream of traffic is interrupted between times $t = 0$ and $t = r$ by the effective red period of a traffic signal. At all times, the traffic approaches the signal freely at rate q and speed v, and after time $t = r$ the signal remains green indefinitely.

Show that the trajectory of x_b, the back of the queue of stationary traffic, initially satisfies

$$x_b = \frac{-qlvt}{(ql - v)},$$

where l is the effective length of a queued vehicle.

Show that the flow $q_r(k)$ past a wave of density k satisfies

$$q_r(k) = -k^2 \frac{dv}{dk}.$$

Using the variable s, the saturation flow, derive an expression for the time at which traffic conditions at the stop-line return to normal.

4 (b) At a signal-controlled road junction there are two streams of traffic, each having green in one of the two stages of the signal cycle The cycle time must not exceed c_0 and proportions λ_1 and λ_2 of the cycle are effectively green for Stages 1 and 2 respectively. The lost time per cycle is L, the flow ratios in Streams 1 and 2 are y_1 and y_2 respectively and their maximum acceptable degrees of saturation are p_1 and p_2 respectively. No minimum green constraints are imposed.

The arrival rates in the two streams are multiplied by a common factor μ. Derive the equations for the three planes in $(\lambda_1, \lambda_2, \mu)$ space that form (together with the plane $\mu = 0$) the boundaries of the region containing acceptable values of $(\lambda_1, \lambda_2, \mu)$.

Hence find the coordinates of the vertex of this region at which μ is largest.

Corresponding to this vertex, what names are given to the signal timings and the conditions under which the junction is operating?

5 At a signal-controlled road junction there are m stages in the signal cycle and n streams of traffic. For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, the effective green times for Stage i and Stream j form proportions λ_i and Λ_j of the cycle respectively. The lost time forms a proportion λ_0 of the cycle, and $\lambda = (\lambda_0, \lambda_1, \lambda_2 \ldots, \lambda_m)$.

The Λ_j are known linear combinations of the components of λ, and the flow ratio of each stream j has a known value y_j.

What is the value of the sum of the components of λ and why is this so?

Express as linear inequalities in the components of λ the usual practical constraints on the cycle time and the durations of the stages.

Provided that $\Lambda_j > y_j$ for all j, the average delay per unit time to vehicles at the junction is approximately proportional to

$$D(\lambda) = \Sigma_j \{f_j(\Lambda_j)/\lambda_0 + g_j(\Lambda_j)\},$$

where $f_j(\Lambda) = Lq_j(1-\Lambda)^2/2(1-y_j)$ and $g_j(\Lambda) = y_j^2/2\Lambda(\Lambda - y_j)$.

S is the set of values of λ such that the above constraints on the components of λ and on the Λ_j are satisfied. One member λ^* of S is known.

Show that there is a member $\hat{\lambda}$ of S such that $D(\lambda) \geq D(\hat{\lambda})$ for all λ in S.

For a junction at which $m = 2$, use the equation for the sum of the components of λ to express the cycle time and stage duration constraints in terms of λ_1 and λ_2 only. Hence sketch the boundaries that S would have in the (λ_1, λ_2) plane if traffic was very light in every stream.

Stream 1 has green only in Stage 1 and none of the lost time is effectively green for this stream. The traffic in Stream 1 is heavy enough to add another boundary to S. Add to your sketch a line representing the constraint $\Lambda_1 > y_1$.