# UNIVERSITY COLLEGE LONDON

University of London

## EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

*B.Sc.* *M.Sci.*

**Mathematics C327: Real Analysis**

COURSE CODE : **MATHC327**

UNIT VALUE : **0.50**

DATE : **17-MAY-06**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

06-C1008-3-40


**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. Define the notion of Lebesgue measurable subsets of $\mathbb{R}$.

   Show that Borel sets are Lebesgue measurable.

   For any non-negative measurable function $f$ on a measure space define an increasing sequence of non-negative, simple, measurable functions converging to $f$.

   Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Show that $f$ is Lebesgue measurable.

2. Define the integral of real functions on a measure space $(\Omega, \mathcal{F}, \mu)$.

   Show that every integrable function is finite almost everywhere.

   State and prove Hölder's inequality for real functions on $(\Omega, \mathcal{F}, \mu)$.

   Find the largest value of $\int_0^1 g(x)x^3\,dx$ where $g$ runs through Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 g^4(x)\,dx \leq 1$.

3. State the Monotone Convergence Theorem.

   State and prove the Dominated Convergence Theorem. (The Monotone Convergence Theorem may be used without a proof, but the Fatoux Lemma, if used, should be stated and proved as well.)

   Suppose that $\mu$ is a Borel measure on $\mathbb{R}$ such that finite sets have measure zero. Show that for every $f \in L_1(\mu)$ the function

$$F(x) = \int_{(-\infty,x)} f\,d\mu$$

   is continuous on $\mathbb{R}$.

4. State the Fubini Theorem and assuming its validity for non-negative measurable functions prove it for functions whose integral exists.

   Show that the function

$$f(x,y) = \frac{2xy}{1 + x^4 + y^4}$$

   is not $\lambda_2$-integrable.

5. Suppose that $\mu$ is a finite signed measure on a $\sigma$-algebra $\mathcal{F}$ of subsets of $\Omega$ such that $\mu(\Omega) < 0$. Show that there is $E \in \mathcal{F}$ so that $\mu(E) < 0$ and $\mu(F) \leq 0$ whenever $F \in \mathcal{F}$ is a subset of $E$.

   State the Radon-Nikodým Theorem.

   Suppose that $\mu, \nu$ are measures on a $\sigma$-algebra $\mathcal{F}$ of subsets of $\Omega$ such that $\mu(\Omega) = 1$ and $\nu(\Omega) = 2$. Let $f$ be the Radon-Nikodým derivative of $\mu$ with respect to $\mu+\nu$. By considering first the sets $\{x : f(x) \leq 0\}$ and $\{x : f(x) \geq 1\}$, find $\mu(\{x : f(x) > 0\})$ and $\nu(\{x : f(x) < 1\})$.