UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C327: Real Analysis

COURSE CODE	:	**MATHC327**
UNIT VALUE	:	**0.50**
DATE	:	**20-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0922-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Define the notions of outer measure and of measure. Prove that the restriction of an outer measure μ^* to the class of μ^*-measurable sets is a measure (check that this class forms a σ-algebra).

   For a subset A of $\mathbb{R}$ let $\nu(A)$ be equal 1 if A is non-empty, and 0 if A is empty.

   Is ν a measure on the σ-algebra of all subsets of $\mathbb{R}$? Show that ν is an outer measure on $\mathbb{R}$, and find σ-algebra of ν-measurable sets.

2. State and prove the Monotone Convergence Theorem for non-decreasing sequences of positive functions. State the Dominated Convergence Theorem.

   Show that
$$\int_0^1 \cos(x^2)\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(2k)!}.$$

3. For given finite measure spaces (Ω, Σ, μ) and (Ω', Σ', μ') define the product σ-algebra $\mathcal{F}$ on $\Omega \times \Omega'$ and the product measure ν on $\mathcal{F}$. State Fubini's Theorem for positive functions and give a proof for characteristic functions.

   For every $a \in \mathbb{R}$, find $\int_E e^{ax+y}\,d\lambda_2(x,y)$ where $E = \{(x,y) \in \mathbb{R}^2 \colon 0 \le x, y;\ x+y \le 1\}$.

4. Define the notion of absolute continuity of measures. State the Lebesgue Decomposition Theorem.

   For a triple of measures μ, ν, η on (Ω, Σ) such that $\mu \ll \nu$ and $\nu \ll \eta$ prove that $\mu \ll \eta$. Find $\frac{d\eta}{d\mu}$ if $\frac{d\eta}{d\nu} = f$ and $\frac{d\nu}{d\mu} = g$. Explain your answer.

   If $\{\mu_n\}$ is a sequence of measures on (Ω, Σ) with $\mu_n(\Omega) \le 1$ show that the formula $\lambda(E) = \sum_{n=1}^{\infty} \frac{1}{n^2}\mu_n(E)$ defines a measure, and every μ_n, $n = 1, 2, 3, \ldots$ is absolutely continuous with respect to λ.

5. Define the Lebesgue spaces $L^p(\mu)$ for $1 \le p \le \infty$. State the Hölder inequality. State the triangle inequality.

   (a) For $f, g \in L^p(\mu)$, $2 \le p \le \infty$ check that their product $f \cdot g$ belongs to $L^{p/2}(\mu)$.

   (b) Using the identity
$$ab - cd = a(b-d) + (a-c)d,$$
   prove that if $f_n, g_n \in L^p(\mu)$, $2 \le p < \infty$, and $\|f_n - f\|_p \to 0$, $\|g_n - g\|_p \to 0$, then
$$\|f_n g_n - fg\|_{p/2} \to 0.$$