# UNIVERSITY COLLEGE LONDON

University of London

## EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

*B.Sc.* *M.Sci.*

**Mathematics C327: Real Analysis**

| | | |
|---|---|---|
| COURSE CODE | : | **MATHC327** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **29-APR-04** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 Hours** |

04-C0919-3-50


**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. Define the Lebesgue outer measure $\lambda^*$ of subsets of $\mathbb{R}$.

   Define the notion of Borel subsets of $\mathbb{R}$.

   State what is meant by saying that a set $S \subset \mathbb{R}$ is Lebesgue measurable.

   Show that for every set $S \subset \mathbb{R}$ there is a Borel set $B \supset S$ such that $\lambda^*(B) = \lambda^*(S)$.

   State results allowing you to deduce that Borel sets are Lebesgue measurable.

   Assume that a set $S \subset \mathbb{R}$ has the property that $\lambda^*(B \setminus S) = 0$ for some Borel set $B \supset S$. Show that $S$ is Lebesgue measurable. (You may assume any general properties concerning Lebesgue measurability but any that you use should be stated.)

2. State the property of continuity of measure.

   State and prove the Monotone Convergence Theorem for non-decreasing sequences of positive functions.

   Prove that $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$ for non-negative measurable functions $f, g$ on $\Omega$.

   Show that
$$\int_\pi^\infty \frac{|\cos(x)|}{x} d\lambda(x) = \infty.$$

3. Explain what is meant by product of two measure spaces.

   State a result guaranteeing existence and uniqueness of the product measure. Prove it for the case of finite measures. Auxiliary results such as the Monotone Class Theorem or Hopf's Theorem (or analogous results if you use different arguments than those in the lecture) may be used without a proof, but should be fully stated.

   Show that the set
$$\left\{(x, y, z) \in \mathbb{R}^3 : x^2 e^{y^2 - z^2} = 1\right\}$$
   is measurable with respect to the 3-dimensional Lebesgue measure $\lambda_3$ in $\mathbb{R}^3$ and that
$$\lambda_3 \left\{(x, y, z) \in \mathbb{R}^3 : x^2 e^{y^2 - z^2} = 1\right\} = 0.$$

4. Let $\mathcal{F}$ be a set of $\mu$-measurable functions. State what is meant by saying that $f \in \mathcal{F}$ is maximal with respect to the order '$\phi \leq \psi$ almost everywhere'.

   Find a maximal element of the set $\mathcal{F}$ of indicator functions of finite subsets of $\mathbb{R}$ with respect to the order '$\phi \leq \psi$ $\lambda$-almost everywhere'.

   State the Radon-Nikodým Theorem. Prove it for the case of finite measures. Auxiliary results such as existence of maximal elements in the order '$\phi \leq \psi$ almost everywhere' (or analogous results if you use a different proof than in the lecture) may be used without a proof, but should be fully stated.

5. Define the Lebesgue spaces $L^p(\mu)$ for $1 \leq p \leq \infty$.

   Let $f_n, f \in L^p(\mu)$. What does it mean that $f_n \to f$ in $L^p(\mu)$?

   Let $S_n$ be a decreasing sequence of $\mu$-measurable sets of finite measure and let $S = \bigcap_{n=1}^{\infty} S_n$. Show that the indicator functions of $S_n$ converge to the indicator function of $S$ in $L^p(\mu)$ when $1 \leq p < \infty$, but not necessarily when $p = \infty$.

   Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

   Find a simple function $g$ on $\mathbb{R}$ so that

$$\|f - g\|_{L^2(\lambda)} \leq \frac{1}{2}\|f\|_{L^2(\lambda)}.$$