UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C327: Real Analysis

COURSE CODE	:	**MATHC327**
UNIT VALUE	:	**0.50**
DATE	:	**21-MAY-03**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

03-C0911-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Define what it means for a real-valued function f on a measure space $(\Omega, \mathcal{F}, \mu)$ to be *measurable.*

   Prove that f is measurable if, for every real c, the set

   $$\{x : f(x) > c\}$$

   is in $\mathcal{F}$.

   Prove that the sum of two measurable functions is measurable.

   Now for a measurable f and t real, let

   $$L_t = \{x : f(x) > t\}$$

   and define $m : \mathbb{R} \to \mathbb{R}$ by

   $$m(t) = \mu(L_t).$$

   Show that m is measurable. (Hint: what happens to $m(t)$ as t increases?)

2. *State* the Monotone Convergence Theorem and Fatou's Lemma.

   State and prove the Dominated Convergence Theorem.

   Show that

   $$\sum_0^\infty \frac{(-1)^n}{2n+1} = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}.$$

3. Define the terms *outer measure* and μ^**-measurability.*

State Caratheodory's restriction theorem.

Let μ be a measure on an algebra $\mathcal{A}$ of subsets of a set Ω. For $E \subset \Omega$ define

$$\mu^*(E) = \inf \left\{ \sum_1^\infty \mu(A_i) : E \subset \bigcup A_i, A_i \in \mathcal{A} \right\}.$$

Show that μ^* is an outer measure and that the sets in $\mathcal{A}$ are μ^*-measurable. Explain briefly what more is needed to conclude that the restriction of μ^* to $\sigma(\mathcal{A})$ is an extension of μ to the generated σ-algebra?

4. *State* the Monotone Class Theorem.

State and prove Fubini's Theorem for non-negative measurable functions on a product space. (You may assume the measurability of sections and marginals of such functions.)

5. State and prove Hölder's inequality for real functions on a measure space $(\Omega, \mathcal{F}, \mu)$.

Deduce the triangle inequality for functions in $L_p(\mu)$, $1 \leqslant p < \infty$.

Show that if f and g are functions in $L_2(\mu)$ then

$$\left\| \frac{f+g}{2} \right\|_2^2 + \left\| \frac{f-g}{2} \right\|_2^2 = \frac{\|f\|_2^2 + \|g\|_2^2}{2}.$$

Deduce that if $\|f\|_2 = \|g\|_2 = 1$ and $\|f - g\|_2 = t$ then

$$\left\| \frac{f+g}{2} \right\|_2 \leqslant 1 - t^2/8.$$