# UNIVERSITY COLLEGE LONDON

*University of London*

# EXAMINATION FOR INTERNAL STUDENTS

*For the following qualifications :-*

*B.Sc.*   *M.Sci.*

**Mathematics C327: Real Analysis**

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| COURSE CODE | : | **MATHC327** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **14-MAY-02** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 hours** |

02-C0910-3-60



**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. Explain what it means for a function defined on the family of subsets of a set $\Omega$ to be an outer measure.

   Define the Lebesgue outer measure $\lambda^*$ and prove that it *is* an outer measure.

   For an outer measure $\mu^*$ define the notion of a $\mu^*$-measurable set.

   *State* Caratheodory's Restriction Theorem.

   Prove that Borel subsets of the real line are $\lambda^*$-measurable.

   What further fact, beyond those described above, do you need, in order to demonstrate the existence of Lebesgue measure on the Borel subsets of the real line?

2. *State* the continuity of measure principle.

   State and prove the Monotone Convergence Theorem. (You may assume superadditivity of the integral.)

3. Let $p$ be the polynomial function given by

$$p(x) = \sum_{0}^{k} a_i x^i.$$

   Prove that for every positive $y$ and natural number $n$,

$$|p(y/n)| \leqslant \sum_{0}^{k} |a_i| y^i.$$

   *State* the Dominated Convergence Theorem.

   Show that as $n \to \infty$

$$n \int_0^\infty p(x)e^{-nx}\, dx \to p(0).$$

4. *State* Fubini's Theorem for non-negative measurable functions on a product space.

   Explain briefly the main ingredients in a proof of this theorem for the indicator function $\mathbf{1}_A$ of a set $A$ in the product $\sigma$-algebra.

   Let $\mu$ be counting measure on the interval $[0, 1]$: so the measure of each subset $E$ of the interval is just the number of points in $E$. Let $\lambda$ be Lebesgue measure on the interval $[0, 1]$. Find a measurable set $A \subset [0, 1] \times [0, 1]$ whose indicator function $f = 1_A$ satisfies

   $$\int f(x, y)\, d\mu(x) = 1$$

   for every $y$, but

   $$\int f(x, y)\, d\lambda(y) = 0$$

   for every $x$.

   Which hypothesis of Fubini's Theorem is false for this example?

5. State and prove the Hahn-Jordan decomposition Theorem. (You may assume that finite signed measures are bounded.)

   Explain what is meant by the term *absolutely continuous* as applied to measures.

   *State* the Radon-Nikodým Theorem.