All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Evaluate the indefinite integrals

$$\text{(i)} \quad \int x(2-x)^6 \, \mathrm{d}x, \qquad \text{(ii)} \quad \int e^x \sin x \, \mathrm{d}x.$$

   (b) Find A, B, C, D such that

$$\frac{2x^4 - x^3 + x^2 + x + 1}{x(1+x^2)(x-1)^2} = \frac{A}{x} + \frac{Bx + C}{1+x^2} + \frac{D}{(x-1)^2}.$$

   Hence, or otherwise, find

$$\int_2^3 \frac{2x^4 - x^3 + x^2 + x + 1}{x(1+x^2)(x-1)^2} \, \mathrm{d}x.$$

2. Consider the function

$$y(x) = \frac{x^3}{x-4}.$$

   (a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

   (b) Find the turning points of y and determine whether they are maxima, minima or points of inflexion.

   (c) Find all asymptotes of the curve $y(x)$.

   (d) Sketch the function $y(x)$ being careful to show any turning points and asymptotes.

3. (a) Let $z = x + iy$ where i is the complex number $i = \sqrt{-1}$. Define $\bar{z}$, the complex conjugate of z, and $|z|$ the modulus of z.

(b) When $x = -\frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$ find:

$$\text{(i) } \bar{z}, \quad \text{(ii) } |z|^2, \quad \text{(iii) } 1 + z + z^2.$$

(c) Let $w = \cos\theta + i\sin\theta$. Show that (i) $w + \bar{w} = 2\cos\theta$, (ii) $w - \bar{w} = 2i\sin\theta$ and (iii) $\bar{w} = 1/w$.

(d) State De Moivre's Theorem and show that $w^k + \bar{w}^k = 2\cos(k\theta)$ for k integer.

(e) Hence, or otherwise, show that

$$\cos^4\theta = \frac{1}{8}(\cos(4\theta) + 4\cos(2\theta) + 3),$$

and find a similar expression for $\sin^4\theta$ in terms of $\cos(4\theta)$ and $\cos(2\theta)$.

4. (a) Using the Ratio Test, the Comparison Test, or otherwise, establish which of the following series converge:

$$\text{(i) } \sum_{n=1}^{\infty} \frac{n+3}{(n+2)^2}, \quad \text{(ii) } \sum_{n=1}^{\infty} \frac{n!(n+2)!}{(2n)!}.$$

(b) For $|x| < 1$ find the Maclaurin series expansions for up to and including x^5 for the functions

$$\text{(i) } x\log_e(1-x), \quad \text{(ii) } \tan^{-1} x.$$

(Hint: In (b) part (ii) you might first consider an integral.)

5. (a) Find the general solution of the differential equation

$$xy^2\frac{dy}{dx} = x^3 + y^3.$$

Find the solution that satisfies $y(1) = 0$.

(b) Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{xy}{1+x^2} = x.$$

6. (a) Find inverse of the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

using the method of cofactors and verify your result.

(b) Find the values of α for which the system

$$\begin{aligned} x + 2y &= 1 \\ x - y - z &= 2 \\ 3y + \alpha z &= -1. \end{aligned}$$

has a unique solution. Find all solutions when $\alpha = 1$.

7. (a) Let $A = \begin{pmatrix} -2 & 4 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 3 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$.

Evaluate the following matrix multiplications: (i) AB, (ii) BC and (iii) A^2.

(b) Find all the eigenvalues of the matrices

$$\text{(i)} \begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

$$\text{(iii) } A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

(c) Find an eigenvector for the smallest eigenvalue of the matrix A in (b) part (iii).