UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. M.Sci.

Mathematics B3C: Pure Mathematics

COURSE CODE	:	**MATHB03C**
UNIT VALUE	:	**0.50**
DATE	:	**09-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0900-3-60

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) By expressing
$$\frac{1+2x-x^2}{x-x^3} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$
for suitable constants A, B, C, which you should find, evaluate
$$\int_{-2}^{2} \frac{1+2x-x^2}{x-x^3}\,\mathrm{d}x.$$

   (b) Evaluate the indefinite integrals
$$\text{(i)} \int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x \qquad \text{(ii)} \int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x.$$

2. Let $z_1 = 2+3i$ and $z_2 = 1+i$, where i is the complex number $\sqrt{-1}$.

   (a) Find, in the form $a+ib$ (a, b real numbers),
$$\text{(i) } z_1+z_2, \quad \text{(ii) } z_1 z_2, \quad \text{(iii) } z_1/z_2.$$

   Now let $z = \cos\theta + i\sin\theta$.

   (b) Show that
$$z^2 = \cos 2\theta + i \sin 2\theta,$$
and state the corresponding formula for z^n for n a positive integer.

   (c) When $\theta = \pi/4$ radians, draw the complex numbers $z, z^2, z^3, z^4, z^5, z^6, z^7, z^8$ on the Argand diagram.

   (d) Find all the solutions of $z^4+1=0$, explaining how you arrive at your answer.

3. Let $\alpha \neq 0$ and

$$A = \begin{pmatrix} 1 & \alpha & 1 \\ 3 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}.$$

(a) Find the determinant of A in terms of α.

(b) Find the nine cofactors of A.

(c) Hence show that the inverse of A is

$$A^{-1} = \frac{1}{\alpha} \begin{pmatrix} -1 & 3\alpha - 1 & 1 - 2\alpha \\ 1 & 1 & -1 \\ 1 & 1 - 4\alpha & 3\alpha - 1 \end{pmatrix} \quad (\alpha \neq 0).$$

(d) Using part (c) with α chosen appropriately, or otherwise, find the solution to the system

$$\begin{aligned} 2x + 2y + 2z &= 4 \\ 3x + y + 2z &= 0 \\ 4x + y + 3z &= 2. \end{aligned}$$

4. Consider the function

$$y(x) = \frac{x^3}{x^2 - 1}.$$

(a) Find dy/dx.

(b) Find the turning points of y and determine whether they are maxima, minima or points of inflexion.

(c) Show that

$$\frac{dy}{dx} \to 1 \text{ as } |x| \to \infty.$$

(d) Sketch the function y being careful to show any turning points and asymptotes.

5. (a) Find, by a suitable change of variables, the general solution of the *homogeneous* differential equation

$$x^2\frac{dy}{dx} = xy - y^2.$$

(b) Find the solution of the differential equation

$$\frac{dy}{dx} - y = x$$

which satisfies $y = 1$ when $x = 0$.

6. Let $a_0, a_1, a_2, \ldots$ be real numbers and consider the series $S = \sum_{k=0}^{\infty} a_k$.

(a) What does it mean to say that $S = \sum_{k=0}^{\infty} a_k$ converges?

(b) Show that

$$1 + x + x^2 + x^3 + \cdots x^{n-1} = \frac{1 - x^n}{1 - x},$$

and hence find the range of values of x for which the series

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots$$

converges, and state its value in terms of x.

(c) Show that for $|x| < 1$

$$-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^k}{k} - \cdots = \log(1 - x).$$

(d) Evaluate the sum

$$\frac{1}{4} - \frac{1}{2(4^2)} + \frac{1}{3(4^3)} + \cdots \frac{(-1)^{k+1}}{k(4^k)} + \cdots$$

7. The function $f(x)$ has the following Taylor series expansion

$$f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

which you are told is convergent for all x.

(a) Write down a formula for the a_k in terms of f and its derivatives and valid for $k = 0, 1, 2, \ldots$.

(b) Find the Taylor series expansions for f up to and including x^5 for the cases

$$\text{(ii) } f(x) = (1-x)^{-2}, \quad \text{(ii) } f(x) = \cos(x^2).$$

(c) Let f, g be functions such that $f(0) = 0 = g(0)$ and $f'(0)$, $g'(0)$ are not both zero. What does L'Hôpital's rule say about the limit

$$\lim_{y \to 0} \frac{f(y)}{g(y)} \ ?$$

(d) Find the following limits:

$$\text{(i) } \lim_{x \to 0} \frac{e^x - 1}{x}, \quad \text{(ii) } \lim_{x \to \pi} \frac{\sin(x)}{x - \pi}, \quad \text{(iii) } \lim_{x \to 0} \frac{x^4}{1 - \cos(x^2)}.$$