UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *B.Sc.(Econ)M.Sci.*

Mathematics M252: Probability and Statistics

COURSE CODE : **MATHM252**

UNIT VALUE : **0.50**

DATE : **23-MAY-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator **is** *permitted in this examination.*

New Cambridge Statistical Tables are provided.

1. (a) The functions given below are zero outside the range specified. In each case, determine whether there is a value of the constant c for which the function is a valid probability density function. If there is, give it, and find the corresponding distribution function.

   (i) $f(x) = c\,(1 - x^2)$ for $-1 < x < 1$.

   (ii) $f(x) = c\,(0.5 - x^2)$ for $-1 < x < 1$.

   (iii) $f(x) = c/\,(1 - x^2)$ for $-1 < x < 1$.

   (b) X is a continuous random variable with probability density function

$$f(x) = \begin{cases} a + bx^2 & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

   If $E(X) = 3/5$, find a and b.

2. A light switch operates by pulling a cord, both to switch the light on and then to switch it off again. The switch will jam with probability p, independently at each pull of the cord. If the switch does not jam, the pull is said to be 'successful'. If the switch jams when the cord is pulled, the state of the light (on/off) is unchanged.

   (a) Let X be the number of successful pulls before the switch jams. Find the probability mass function of X.

   (b) Assuming that the light is off initially, show that when the switch jams, the light is off with probability $1/(2 - p)$.

   (c) Find the probability generating function of X. Hence, or otherwise, find the expected value of X.

   (d) Find the distribution function of X.

3. In this question, Φ denotes the distribution function of the standard normal distribution, and ϕ denotes its density.

The lifetime of a certain type of car tyre (in thousands of kilometres) is normally distributed with mean 34 and standard deviation 4.

(a) (i) What is the probability that such a tyre lasts over 40,000 kilometres?

(ii) What is the probability that such a tyre lasts between 30,000 and 35,000 kilometres?

(iii) If such a tyre has already survived 30,000 kilometres, what is the probability that it survives another 10,000 kilometres?

(b) A 4-wheeled car is fitted with 4 new tyres. Assume that the lifetimes of all tyres are independent, with distribution as above.

(i) Write down an expression for the probability that no tyres fail within the first x kilometres. Leave your answer in terms of Φ.

(ii) Find an expression for the density of the time until the first tyre failure. Leave your answer in terms of Φ and ϕ.

4. (a) A continuous random variable X, taking non-negative values, has probability density function

$$f(x) = \frac{(\eta x)^{\nu}}{x\Gamma(\nu)} e^{-\eta x} \qquad (x > 0),$$

for parameters $\nu > 0$ and $\eta > 0$, where $\Gamma(.)$ denotes the gamma function.

(i) Name this distribution, and state its mean and variance.

(ii) Show that the moment generating function (MGF) of this distribution is

$$M(t) = \left[1 - \frac{t}{\eta}\right]^{-\nu}.$$

Use this to verify the mean and variance of the distribution.

(b) Suppose now that $X_1, \ldots, X_n$ are independent random variables, such that X_i has the distribution given above with parameters ν_i and η_i. Define $S = \sum_{i=1}^{n} X_i$. Write down an expression for the MGF of S. State any restrictions on the parameters $\{\nu_i, \lambda_i : i = 1, \ldots, n\}$ under which the distribution of S has the same form as that of X. In this case, state the parameters of the distribution of S.

(You may use, without proof, the result that the MGF of a sum of independent random variables is the product of their individual MGFs).

5. In a study to investigate growth rates of trees, 20 seedlings were grown in a greenhouse. Half of them received a dose of nitrogen at planting time; the remainder received no nitrogen. The seedlings were all of the same type and, apart from the nitrogen dose, were planted in identical conditions. The stem weights, recorded in grams, at the end of 140 days, were recorded as follows:

No nitrogen	0.32	0.53	0.28	0.37	0.47	0.43	0.36	0.42	0.38	0.43
Nitrogen	0.36	0.43	0.47	0.49	0.52	0.75	0.69	0.66	0.62	0.46

Each group of observations is assumed to be drawn from a normal distribution.

(a) Calculate the sample mean and standard deviation for each group of seedlings.

(b) Test, at the 5% level, the hypothesis that the two sets of observations come from distributions with the same variance. State your conclusions clearly.

(c) Assuming that the underlying variances are the same in each group, construct a 95% confidence interval for the difference in the mean stem weights. Comment on the result.

6. (a) Suppose that observations $X_1, \ldots, X_n$ are drawn from a probability distribution with unknown parameter θ, and that $T = T(X_1, \ldots, X_n)$ is an estimator of θ. Define the mean squared error (MSE) of T as an estimator of θ, and show that the MSE can be written as

$$MSE_\theta(T) = b_\theta^2(T) + \mathrm{Var}_\theta(T) \ ,$$

where $b_\theta(T)$ is the bias of T and $\mathrm{Var}_\theta(T)$ is its variance.

(b) Under normal conditions, when a fused electrical appliance is switched on the fuse fails with probability p, independently from one occasion to the next. A fuse manufacturer wishes to estimate the mean number of power cycles up to and including fuse failure. He notes that the expected number of power cycles up to and including the first failure is $1/p$. To estimate this quantity he proposes to take a large number, n, of fuses, install each of them in identical appliances, switch each appliance on once and record X, the number of fuse failures. He then proposes to use $T = n/X$ as an estimator of $1/p$.

(i) Show that $T^{-1} = X/n$ is an unbiased estimator of p. What is the MSE of T^{-1} as an estimator of p?

(ii) By considering the possible values taken by T, or otherwise, show that T is a biased estimator of $1/p$. What is the MSE of T as an estimator of $1/p$?

(iii) Suggest an alternative experiment that would allow the construction of an unbiased estimator of $1/p$.