UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc.

Mathematics M310: Operational Research and Mathematical Modelling

COURSE CODE	:	**MATHM310**
UNIT VALUE	:	**0.50**
DATE	:	**03-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0956-3-30

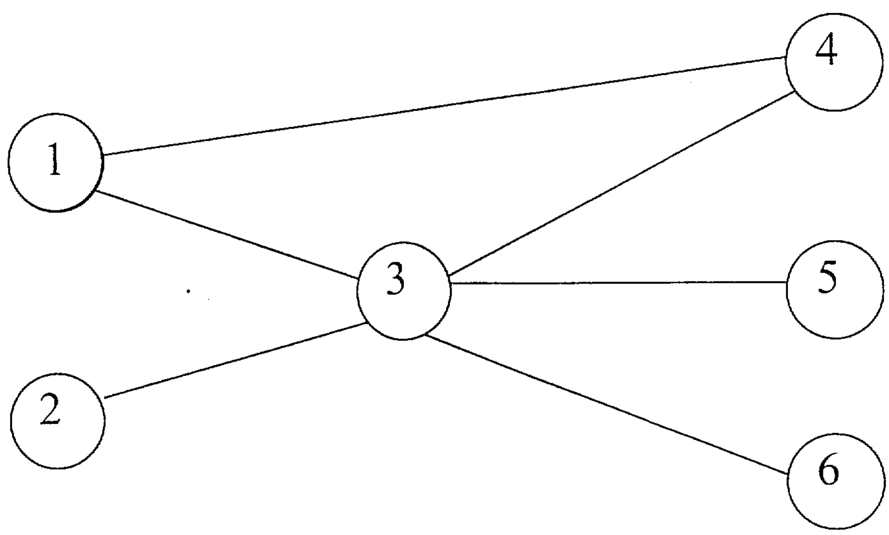
All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Suppose $f : \mathbf{R}^2 \to \mathbf{R}^1$ is a function defined by $f(x_1, x_2) = x_1^2 + 3x_2^2 + 2x_1$.

If the initial point $\mathbf{x}^0 = (0, 1)^T$ is used as a starting point for the method of steepest descent to minimise f, implement the algorithm to determine the next point $\mathbf{x}^1$.

(b) A wood working company owns two factories which must suppply three funeral directors with coffins using the distribution network shown in the diagram below. The factories are at locations 1 and 2 and the funeral directors are at locations 4, 5 and 6. Node 3 represents an intermediate town.

The factory at location 1 can produce at most $P1$ coffins per week and the factory at location 2 can produce at most $P2$. The funeral directors at locations 4, 5 and 6 have a contract specifying that $D4$, $D5$ and $D6$ coffins should be supplied each week, respectively. It is possible to transport a maximum of M_{ij} coffins per week along link (i, j) at a cost of C_{ij} per coffin. The cost to the company of producing coffins is the same at both factories.



Derive an integer programming problem, the solution of which could be used by the wood working company to choose the number of coffins x_{ij} that are transported on each link (i, j) per week in order to minimise the weekly cost of transporting the coffins.

2. (a) Suppose $f : \mathbf{R}^n \to \mathbf{R}^1$ is a function whose domain of definition is the whole of $\mathbf{R}^n$.

What condition must be satisfied if f is a convex function?

(b) Suppose $f : \mathbf{R}^2 \to \mathbf{R}^1$ is a function defined by $f(x_1, x_2) = \min\{x_1 - 1,\ x_2 - 1\}$

i) Show that f is a concave function.

ii) Sketch the level sets of f indicating the direction in which the function increases.

iii) Hence, or otherwise, find the maximum value of f that occurs for values of x_1 and x_2 with $x_1^2 + x_2^2 \leq 4$.

3. (a) If P is the transition matrix for a finite, irreducible Markov Chain with m states and there exists a matrix Π such that $P^n \to \Pi$ as $n \to \infty$ then show that the elements of Π are given by

$$\pi_{ij} = \sum_{k=1}^{m} \pi_{ik} p_{kj}$$

where p_{ij} and π_{ij} denote the elements in the i^{th} row and j^{th} column of P and Π respectively.

State, without proof, the relationship between the rows of Π.

(b) A consultant has been asked to evaluate a potential penalty system for drivers who persistently commit speeding offences. The system is based on assigning drivers to successively harsher penalty categories 1, 2, 3 or 4, with higher fines for the more frequent and persistent offenders. The system also allows infrequent offenders to benefit from reduced penalties. The consultant models this process using a Markov chain to represent the annual transitions between categories with the following transition matrix:

$$Q = \begin{array}{cc} & \text{To} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} \text{From } 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1-p & p & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 0 & 1-q & q \\ 0 & 0 & 1-r & r \end{bmatrix} \end{array}$$

It can be assumed that $0 < p, q, r < 1$.

i) Classify each state as transient or persistent.

ii) Using the results from (a) or otherwise, find the limit of Q^n as n tends to infinity.

4. (a) A single server queue has exponential service times with mean $1/\mu$, and Poisson arrival rates λ. The number in the system (i.e. in the queue or being served) is governed by the differential equations:

$$\begin{aligned} P_0'(t) &= -\lambda P_0(t) + \mu P_1(t) \\ P_n'(t) &= \lambda P_{n-1}(t) - (\lambda + \mu) P_n(t) + \mu P_{n+1}(t) \qquad \text{for } n > 0 \end{aligned}$$

where $P_n(t)$ denotes the probability that there are n in the system at time t.

i) Show that if a steady state solution exists then

$$\pi_n = \rho \pi_{n-1}$$

where π_n denotes the probability of there being n in the system in the steady state, and $\rho = \dfrac{\lambda}{\mu}$.

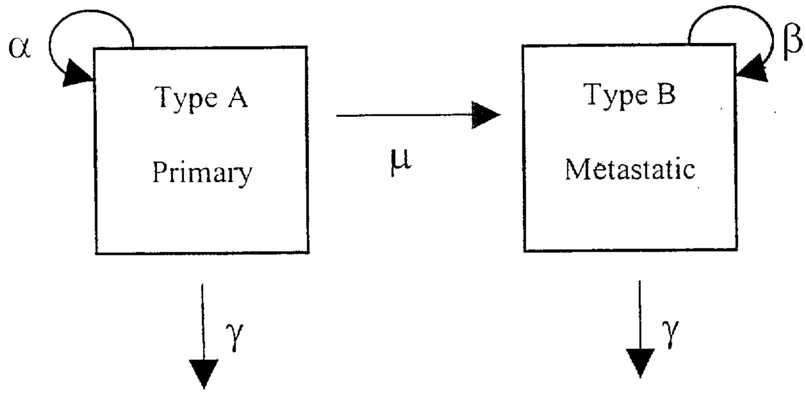
ii) Find the steady state distribution and state the condition under which it exists.

(b) A cobbler operates a 'while-you-wait' shoe repair service. Each repair job is carried out by the cobbler on a first-come-first-served basis. Repair times are exponentially distributed with mean 1/3 hour, and arrival rates are Poisson with mean 2 per hour.

i) What proportion of the time is the cobbler busy repairing shoes assuming the system is in steady state?

ii) When a new customer arrives, what is the expected number of repairs that have to be finished before she can be served?

5. (a) A tumour consists of two types of cell, primary (type A) and metastatic (type B). Type A cells become type B cells as a first order process with transfer coefficient μ, independent of the amount of type B cells present. The death of both types of cell is a first order process with coefficient γ. Both types of cell reproduce as a first order process, type A cells with coefficient α and type B cells with coefficient β. If the tumour remained untreated, the number of both types of cell would increase.



Write down the differential equations that govern the amounts of type A and type B cells.

(b) The most suitable drug treatment for the tumour depends on the relative proportions of type A and type B cells.

Show that, if $(\alpha - \mu) > \beta$, the ratio of the numbers of type A cells to type B cells tends to

$$1 : \frac{\mu}{(\alpha - \mu - \beta))}$$

as time increases.

(c) The doctors treating the tumour select a combination of drugs that increases the elimination coefficient γ for both types of cell by a facctor w so that the elimination coefficient becomes $w\gamma$.

Show that if the tumour is to decrease in size then we must have

$$w > \frac{(\alpha - \mu)}{\gamma}.$$