UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

M.Sc. *PG Dip*

Nonlinear Systems

COURSE CODE : **MATHGM02**

DATE : **05-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-N0132-3-40

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. Write down, without proof, Hamilton's equations of motion for a conservative mechanical system with a vector of generalized coordinates $\mathbf{q} = (q_1, ..., q_n)$ and generalized momenta, $\mathbf{p} = (p_1, ..., p_n)$.

   Show that if the Hamiltonian is translationally invariant in a certain direction in configuration space $\mathbf{q}$ then the component of the momentum vector in this direction is conserved along trajectories of the system's motion.

   A particle of mass m is in a one-dimensional motion under the action of a force $F = F(x(t), x'(t), t)$. Define a linear momentum, $p(t)$, and show that Newton's equation of motion gives rise to a system of two first-order equations for the coordinate $x(t)$ and momentum $p(t)$. Show also that the resulting system is not Hamiltonian unless the force F is independent of $x'(t)$.

   Generalize this example to the case of two particles in the following way. Assume that Newton's equations of motion for the particles can be written as

$$\begin{aligned} m_1 x_1''(t) &= F_1(x_1(t), x_2(t), x_1'(t), x_2'(t), t) \\ m_2 x_2''(t) &= F_2(x_1(t), x_2(t), x_1'(t), x_2'(t), t) \end{aligned}$$

   where the coordinates x_1 and x_2 refer to the position of particles with masses m_1 and m_2. Derive the corresponding system of first-order equations for the coordinates x_1, x_2 and momenta p_1, p_2. Show that the system obtained is Hamiltonian if the forces F_1, F_2 do not depend on the momenta p_1, p_2 explicitly and that the system Hamiltonian can be written as

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + G(x_1, x_2, t)$$

   for some function G. Find an expression for the forces F_1, F_2 in terms of G. Hence, or otherwise, show that if interaction between the particles is mutual, i.e. $F_1 = -F_2$, then the forces F_1, F_2 depend only on time t and the distance $x_1 - x_2$ between the particles.

2. Let A be a real, non-singular, $n \times n$ matrix with n linearly independent eigenvectors. State a sufficient condition for $\mathbf{x} = \mathbf{0}$ to be an asymptotically stable equilibrium of the linear system,

$$\dot{\mathbf{x}} = A\mathbf{x}.$$

Show that if at least one of the eigenvalues of matrix A has a positive real part then the equilibrium at $\mathbf{x} = \mathbf{0}$ is unstable.

Explain what is meant by a locally asymptotically stable equilibrium of a general n-dimensional dynamical system, $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$.

Find the equilibria of the following two-dimensional system,

$$\begin{aligned} \dot{x} &= ax + y - \frac{a+1}{\mu} \\ \dot{y} &= \frac{1}{x} - \frac{1}{y} \end{aligned}$$

where a and μ are real parameters, $\mu \neq 0$ and $x \neq 0, y \neq 0$. Show that the equilibria are unstable if $a < -1$ for any μ.

Determine the nature of the equilibria of this system in the three cases: $|\mu| = 1, |\mu| > 1$ and $|\mu| < 1$ (examine only generic equilibria).

3. Give an example (with explanation and justification of the local stability properties at equilibria) of a one-dimensional dynamical system, $\dot{x} = f(x, \mu)$, which exhibits a saddle-node bifurcation. Sketch the bifurcation diagram.

Investigate equilibria and determine their local stability and bifurcations for the system,

$$\frac{dx}{dt} = (x-1)^2 (x-2)^2 - a,$$

where a is a real parameter. Sketch a bifurcation diagram.

Investigate equilibria and limit cycles and determine their bifurcations and stability for the following complex dynamical system,

$$\frac{dz}{dt} = z\left[1 + 2i - a|z| + |z|^2\right],$$

with variable real parameter α. Sketch the typical phase portraits in the plane $z = x + iy$.

4. Let $\phi(x) : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable, one-dimensional, discrete dynamical system. Write down a sufficient condition for an equilibrium of $\phi(x)$ to be locally asymptotically stable.

Show that a sufficient condition for a two-cycle, (x_1, x_2), to be stable can be written as $|\phi'(x_1)\,\phi'(x_2)| < 1$.

Let $\phi(x) = x + a\sin x$ with a real parameter a. Determine equilibria and their stability for this map (consider only generic cases).

Show that if the value of the parameter a increases from $a = 0$ then a period doubling bifurcation takes place in the range $0 < x < 2\pi$ at the critical value $a_c = 2$ with a two-cycle appearing at $x \approx \pi \pm \sqrt{3(a-2)}$ when the value of a is close to $a_c = 2$. Determine the stability properties of the two-cycle for small and positive $a - 2$.

5. Show that travelling-wave solutions of the sine-Gordon equation,

$$u_{xx} - u_{tt} = \sin u,$$

are governed by an ordinary differential equation of the form,

$$\frac{1}{2}\left(1-\lambda^2\right)\left(f'\right)^2 = C_1 - \cos f,$$

where $u = f(\eta)$, $\eta = x - \lambda t$ and C_1, λ are constant parameters.

Making a change of variables, $g(\eta) = \tan\left(\frac{1}{4}f(\eta)\right)$, obtain the following equation for $g(\eta)$,

$$8\left(1-\lambda^2\right)\left(g'\right)^2 = \left(C_1+1\right)\left(1+g^2\right)^2 - 2\left(1-g^2\right)^2,$$

and verify that $g(\eta) = C\exp(k\eta)$ is a solution for an arbitrary C but with the values of the constants k and C_1 determined uniquely in terms of λ. Hence, or otherwise, derive a non-linear wave solution of the sine-Gordon equation in the form

$$u = 4\arctan\left[C\exp\left(\pm\frac{\eta}{\sqrt{1-\lambda^2}}\right)\right], \qquad \eta = x - \lambda t.$$