UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C311: Methods Of Mathematical Physics I

COURSE CODE	:	**MATHC311**
UNIT VALUE	:	**0.50**
DATE	:	**29-APR-05**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

05-C0919-3-70

TURN OVER

All questions may be attempted but only marks obtained on the best ***four*** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Determine a function $f(t)$ of the complex variable t and contours C_1 and C_2 so that

$$y_i = \int_{C_i} e^{xt} f(t)\, dt \qquad (i = 1, 2)$$

are non-trivial independent solutions of the differential equation

$$x\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + xy = 0\,. \qquad (x > 0)$$

Show that only one of the solutions is bounded as $x \to 0+$.

2. The equation of motion of a body moving in a straight line is

$$\ddot{x} + f(\dot{x}) + g(x) = 0$$

with $f(x)$ and $g(x)$ regular functions of x and the dot denoting differentiation with respect to the variable t.

Obtain the differential equation in the phase plane and show that

(a) periodic motion of the body occurs if and only if the corresponding phase trajectory is closed,

(b) periodic motion of the body is impossible if $\dot{x}f(\dot{x})$ has constant sign.

The equation of motion of a long straight metal bar, restrained by springs and attracted to a parallel current carrying electrical conductor, is

$$\frac{d^2x}{dt^2} + \lambda\left[8x - \frac{1}{1-x}\right] = 0\,, \qquad (0 < x < 1)$$

where λ is a positive constant. Identify the nature of the singular points and sketch the trajectories in the phase plane.

Deduce that if $dx/dt = 0$ when $x = \frac{1}{2}$, the motion of the metal bar is periodic.

3. Show that the differential equation

$$\frac{d^2x}{dt^2} + \epsilon f(x)\frac{dx}{dt} + x = 0\,, \qquad (\epsilon > 0)$$

where $f(x)$ is an integrable functions of x and ϵ is a constant, can, if y is suitably chosen, be expressed in the form

$$\frac{dy}{dx} = \frac{x}{\epsilon F(x) - y}\,,$$

where $F'(x) = f(x)$.

Assuming that a unique periodic solution $x(t)$ exists with amplitude A and period T for all values of ϵ, show that for a certain closed curve γ in the (x, y) plane,

$$\oint_\gamma y f(x)\, dx = 0\,.$$

If $f(x) = x^2 - 1$, show that

(a) $A = 2$, $\quad T = 2\pi\,,$ $\quad (\epsilon \ll 1)\,,$

(b) $A = 2$, $\quad T = \epsilon[3 - 2\ln 2]\,,$ $\quad (\epsilon \gg 1)\,.$

[You may assume that $\int_0^{2\pi} \sin^2 t\, dt = \pi$ and $\int_0^{2\pi} \sin^2 t \cos^2 t\, dt = \frac{1}{4}\pi$.]

4. Show that the equation

$$\ddot{x} + \epsilon f(x, \dot{x}) + x = 0\,,$$

where ϵ is a positive constant and the dot denotes differentiation with respect to t, possesses a solution of the form $x = A(t)\sin[t + \phi(t)]$ if

$$\begin{aligned} \dot{A} &= -\epsilon f(A\sin\chi, A\cos\chi)\cos\chi\,, \\ \dot{\phi} &= \epsilon A^{-1} f(A\sin\chi, A\cos\chi)\sin\chi\,, \end{aligned}$$

with $\chi = \phi + t$. If $\epsilon \ll 1$, describe the method of slowly varying amplitude and phase for finding approximate solutions of these equations.

For the equation

$$\ddot{x} + \epsilon[\dot{x}^3 - 2\dot{x}] + x = 0\,,$$

where ϵ is a constant such that $0 < \epsilon \ll 1$, use the method to show that the amplitude of the *periodic* solution is $(8/3)^{\frac{1}{2}}$.

Show also that the general solution $x(t)$ is given approximately by

$$x(t) = 2\sqrt{2}A_0\, e^{\epsilon t}\left[3A_0^2 e^{2\epsilon t} - 1\right]^{-\frac{1}{2}} \sin(t + \phi_0)\,,$$

where A_0 and ϕ_0 are constants. Hence deduce that the periodic solution is attained as $t \to \infty$ whatever the value of A_0 and is therefore a limit cycle.

[You may assume that $\int_0^{2\pi} \cos^2\chi\, d\chi = \pi$ and $\int_0^{2\pi} \cos^4\chi\, d\chi = \frac{3}{4}\pi$.]

5. A function $F(t)$ can be expanded in a power series in t for $0 \le t < t_0$ of the form

$$F(t) = \sum_{n=0}^{\infty} a_n t^{\lambda_n}, \qquad (a_0 \neq 0)$$

where the sequence $\{\lambda_n\}$ increases with n and $\lambda_0 > -1$. For $t > t_0$, the function satisfies the inequality

$$|F(t)| < Be^{ct}$$

for some positive constants B, c. Show that as $x \to +\infty$,

$$\int_0^{\infty} e^{-xt} F(t)\, dt \sim \sum_{n=0}^{\infty} \frac{a_n (\lambda_n)!}{x^{\lambda_n + 1}}.$$

Hence show that as $x \to +\infty$:

(a) $$\int_1^{\infty} e^{-xt} \ln \sqrt{t^2+1}\, dt \sim \frac{e^{-x}}{2}\left[\frac{\ln 2}{x} + \frac{1}{x^2} + O\left(\frac{1}{x^3}\right)\right].$$

(b) $$\int_0^{1} t^x \tan^{-1} t\, dt \sim \frac{\pi}{4x} - \left(\frac{\pi}{4} + \frac{1}{2}\right)\frac{1}{x^2} + O\left(\frac{1}{x^3}\right),$$

[You may assume that $\int_0^{\infty} e^{-u} u^{\lambda}\, du = \lambda!$ $\qquad (\lambda > -1)$]

END OF PAPER