UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Eng.* *M.Sci.*

Mathematics B46: Mathematics and Statistics for Computer Scientists

COURSE CODE	:	**MATHB046**
UNIT VALUE	:	**0.50**
DATE	:	**08-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1000-3-80

TURN OVER

All questions may be attempted. Marks will be determined by the best **five** *solutions taken from both Section A and Section B. However, within either section, no more than* **three** *solutions will be considered.*
The use of an electronic calculator is **not** *permitted in this examination.*

Section A

1. You may use the following formulae as necessary:

   An exponentially distributed random variable X with mean $E(X) = 1/\lambda$ has probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   A Poisson distributed random variable Y with mean μ has

$$P(Y = k) = \frac{e^{-\mu}\mu^k}{k!} \text{ for } k = 0, 1, 2, \ldots.$$

   Emails arrive in my mailbox according to a Poisson Process of rate 5 per hour.

   (a) Find the probability that between 4:30pm and 5pm this afternoon, exactly 3 email messages will arrive in my mailbox.

   (b) Let T be the time (in minutes) beween the arrivals of the 7th and the 8th email messages tomorrow. Name the distribution of T and state its mean. Find $P(T \leqslant 10)$.

   (c) The number of emails that arrive each hour is recorded during one weekend (i.e. during 48 time intervals of 1 hour each), and using these data the average number of emails that arrive per hour is calculated. Find approximately the probability that this average is less than 4.7 .

   Suppose that each email that arrives in my mailbox has a probability of 0.4 of being a spam message, independently of all other emails.

   (d) Let N be the number of emails that will arrive tomorrow, up to and including the first spam message. Name the distribution of N and state its mean. Find $P(N = 4)$.

   (e) Find the probability that amongst the next 10 emails that arrive, there will be at least 3 spam messages.

   (f) Find approximately the probability that amongst the next 100 emails that arrive, there will be more than 50 spam messages.

2. Four fair dice are thrown at the same time. The numbers shown on different dice are independent of each other.

   (a) Denote by X the number of 6's obtained. Name the distribution of X. Calculate $P(X \geqslant 2)$.

   (b) Find the probability that the four dice all show the same number.

   (c) Find the probability that the four dice show four different numbers.

   (d) Find the probability that two 1's and two 6's are obtained.

   (e) Find the probability that at least two 1's or at least two 6's are obtained.

   (f) Find the conditional probability that two 1's are obtained, given that at least two 6's are obtained.

   (g) Suppose that the four dice are thrown, simultaneously, until 4 equal numbers are obtained. (At each attempt all four dice are thrown, and different attempts are independent of each other.) Find the probability that fewer than 100 attempts are needed to obtain 4 equal numbers.

3. A factory has two production lines making metal components of identical appearance.

   The proportion of copper in a component made by production line A is a random variable, X, with probability density function given by

$$f(x) = \begin{cases} 6x(1-x) & \text{for } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Calculate the mean and the variance of X.

   (b) Calculate $P(X < 0.3)$.

   The proportion of copper in a component made by production line B is a random variable, Y, with probability density function given by

$$f(x) = \begin{cases} 1 & \text{for } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (c) Name the distribution of Y and state its mean.

   (d) Write down the cumulative distribution function of Y.

   A component is discarded if its proportion of copper is less than 0.3. Of the components produced in the factory, 75 % are from production line A, while the remaining 25 % are from production line B.

   (e) Find the probability that a randomly selected component made in the factory will be discarded.

   (f) A component is chosen at random from those that are discarded. Find the probability that this component was made by production line A.

4. The manufacturer of a certain type of light bulb claims that its mean lifetime is 55 days. A sample of 25 light bulbs of the type concerned are tested to destruction. The following data are obtained:

lifetime (days)	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
frequency	1	0	2	6	6	5	2	2	1

(a) Find the median, the upper and lower quartiles, and the interquartile range of this data set.

(b) Calculate the sample mean, the sample variance and the coefficient of variation.

(c) Perform an appropriate statistical hypothesis test to assess, at a significance level of 5%, whether the mean lifetime of the type of light bulb concerned is in fact shorter than the manufacturer claims. State any assumptions that you make and formulate your conclusion clearly.

Section B: Use a separate answer book for this section

5. (a) Prove, from the definition of the derivative, that if y is the function $y = x^3 + \frac{2}{x}$ then its derivative dy/dx is given by $dy/dx = 3x^2 - \frac{2}{x^2}$.

   (b) Define the function $\arcsin(x)$. Explain why the derivative of $\arcsin(x)$ is $1/\sqrt{1-x^2}$.

   (c) Find the absolute value and the argument of each of the following complex numbers: $-2-i$; e^{5-3i} ; $\frac{1+i}{1-i}$; $\sin(2i)$.

6. (a) Find the following limits:

   (i) $\lim_{x\to 0} \frac{\cos(2x)-1}{\sin^2(3x)}$;

   (ii) $\lim_{x\to 0} \frac{\tan(x)-x}{\sin(2x)-\sin(x)-x}$;

   (iii) $\lim_{x\to 0} \frac{\int_0^x \sin(t^2)dt}{x^3}$.

   (b) Find the Taylor series expansion of $f(x) = \frac{1}{(3-2x^3)}$ at the point $x = 0$.

7. (a) Find all stationary points of the function $f(x,y) = e^{x^2-xy+y}(x-y)$ and classify them as local maxima, local minima or saddle points.

   (b) Find the volume of the 3-dimensional region which consists of the points satisfying the following inequalities: $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le e^{x-y}(x+y)$.

8. Let $f(x) = x(\ln^2(x) - 1)$.

   (i) Find extremal points (i.e. points of local maximum or local minimum) of f.

   (ii) Draw the graph of function f.

   (iii) Find the area of the region located below the x-axis, but above the graph of f.