UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C398: Mathematics in Biology 2

COURSE CODE : **MATHC398**

UNIT VALUE : **0.50**

DATE : **27-MAY-05**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

05-C0946-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Consider the selection model with n alleles $A_1, A_2, \ldots, A_n$ in a large, randomly mating, diploid population:

   a) How do the frequencies $p_1, p_2, \ldots, p_n$ evolve from one generation to the next? Explain the relevant parameters (fitness of a genotype, etc.)

   b) State, without proof, the fundamental theorem of natural selection.

   c) Define *stability* and *asymptotic stability* for fixed points.

   d) How can fixed points and asymptotically stable fixed points be characterized in terms of the mean fitness function?

   e) Consider three alleles A_1, A_2, A_3 where all homozygotes are lethal (ie, $w_{ii} = 0$ for $i = 1, 2, 3$) and heterozygotes have fitnesses $w_{12} = 1, w_{13} = w_{23} = \frac{1}{4}$: Determine all fixed points and their invasion and stability properties.

2. Consider the haploid selection model in discrete time. Let $A_1, A_2, \ldots, A_n$ be the n possible types, and $p_1, p_2, \ldots, p_n$ be their frequencies in a large population. Let $v_i \geqslant 0$ denote the fitness of A_i.

   a) Explain why the frequencies in the next generations are given by

   $$p_i' = \frac{v_i p_i}{\sum_{k=1}^{n} v_k p_k}$$

   b) For $n = 2$, and $v_1 = 1, v_2 = .5$ find a formula for the frequencies after t generations. Determine the limit $t \to \infty$.

   c) Assuming $v_1 > v_2 > \cdots > v_n$, show that only one type survives in the long run. Which one?

   d) Show that mean fitness $V(p) = \sum_{k=1}^{n} v_k p_k$ is monotonically increasing over time: $V(p') \geqslant V(p)$ with equality only if $p = p'$.

   e) Which p maximizes the mean fitness $V(.)$?

3. Consider the selection-mutation model with 2 alleles A_1, A_2 in a large, randomly mating, diploid population:

a) How does the frequency p of allele A_1 evolve from one generation to the next? Explain the relevant parameters (fitness of a genotype, mutation rate, etc.)

b) Show that this is equivalent to the difference equation

$$p' - p = \frac{p(1-p)}{2V(p)} \frac{dV(p)}{dp}$$

with $V(p) = p^{2\nu}(1-p)^{2\mu}\bar{w}^{1-\nu-\mu}$, where $\bar{w}$ is mean fitness and μ, ν are mutation rates.

c) Is there an analogue to the fundamental theorem of natural selection for this model? Explain why each orbit converges to a fixed point.

d) What can be said about the number of fixed points?

e) How can the asymptotically stable fixed points of the selection-mutation map be characterized in terms of the function V?

f) Consider genotypes A_1A_1, A_1A_2, A_2A_2 with fitnesses given by $.0, .5, 1.0$, respectively, and the mutation rate from A_2 to A_1 is a small number ν (and there is no mutation in the other direction). Show that there is a unique fixed point describing selection–mutation balance, and calculate it.

4. Consider a symmetric, two-player game with n strategies labelled $1, 2, \ldots, n$ and payoff matrix $A = (a_{ij})$.

a) Define the following terms associated with this game: strict equilibrium, Nash equilibrium, evolutionarily stable strategy (ESS).

b) What is the logical relation between these equilibrium concepts?

c) Which of these types of equilibria exist in every game?

d) Write down the replicator dynamics. Which of the equilibrium concepts mentioned in a) give rise to asymptotically stable equilibria?

e) Compute all Nash equilibria and ESS for the following Hawk–Dove–Retaliator game:

	H	D	R
H	-1	2	-1
D	0	1	$\frac{2}{3}$
R	-1	$\frac{4}{3}$	1

f) Sketch the phase portrait of the replicator dynamics for this game.

5. Consider an asymmetric, two-player game with $n \times m$ payoff matrices $A = (a_{ij})$ and $B^T = (b_{ij})$.

a) Define the following terms associated with this game: Nash equilibrium, strict equilibrium.

b) Show that a strict equilibrium is a monomorphism (ie, pure strategy).

c) Write down the (standard) replicator dynamics for such games.

d) Show that a strict equilibrium is an asymptotically stable equilibrium of the replicator dynamics.

e) Consider an asymmetric version of the hawk–dove game: The first population are owners of a territory, the second population are intruders. Hawks fight harder when they are owners and win in 2/3 of the contests. This leads to the payoff matrices

	H	D
H	$\frac{2V-C}{3}, \frac{V-2C}{3}$	$V, 0$
D	$0, V$	$\frac{V}{2}, \frac{V}{2}$

where $V > 0$ denote the value of the territory and $-C < 0$ is the cost of injury.

Under which condition on V, C is playing 'Hawk if Owner' a dominant strategy?

f) Assuming $V = C = 1$, determine the Nash equilibria and strict equilibria of this game. Sketch the phase portrait of the replicator dynamics.