UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics B8: Mathematics For Physics And Astronomy

COURSE CODE	:	**MATHB008**
UNIT VALUE	:	**0.50**
DATE	:	**15-MAY-06**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

06-C0996-3-100

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let U be a connected open set in $\mathbb{R}^2$ and f a function defined on U. State what it means to say that f is harmonic on U.
Show that if f is equal to the real part of an analytic function $\widetilde{f}$ on U, then $\widetilde{f}$ is unique up to the addition of a constant.
(b) Let $U = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. For each of the following functions f defined on U, find an analytic function $\widetilde{f}$ such that f is equal to the real part of $\widetilde{f}$:
(i) $f(x, y) = \log(x^2 + y^2)$,
(ii) $f(x, y) = x \cos x \cosh y + y \sin x \sinh y$.

2. (a) State the Cauchy integral formula for an analytic function f defined on a simply connected domain U.
Let z_0 be a point in U and C a simple closed curve passing anti-clockwise around z_0. Show that for every positive integer n, the function

$$g_n(z) = \int_C \frac{f(w)}{(w - z)^n}\, dw,$$

defined for points z inside C, is analytic with derivative $n\, g_{n+1}(z)$. Hence deduce that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw,$$

for all such n.
(b) Show that

$$\frac{1}{2\pi i} \oint_C \frac{e^{tz}}{z^{n+1}}\, dz = \frac{t^n}{n!},$$

where C is the unit circle $\{z : |z| = 1\}$.

3. (a) State and prove Taylor's theorem for an analytic function f defined on the domain U.

(b) Find the first three non-zero terms of the Taylor series of $z \cot z$ on the disk $\{z \in \mathbb{C} : |z| < \pi\}$ and state its radius of convergence. Hence, or otherwise, determine the first three non-zero terms of an expansion of $f(z) = z^{-1} \cot (z^{-1})$ on $\{z \in \mathbb{C} : |z| > \pi^{-1}\}$.

4. (a) State the Residue theorem. By making the substitution $z = e^{i\theta}$, or otherwise, prove that

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}.$$

(b) State Jordan's Lemma for a continuous function f defined on the upper half plane $\{z \in \mathbb{C} : \operatorname{Im}(z) \geqslant 0\}$. Show that,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx = \frac{\pi}{a} e^{-a},$$

where a is a positive real number.

5. The curve assumed by a uniform cable suspended between the points $(-1, 0)$ and $(1, 0)$ minimizes the potential energy defined by

$$\int_{-1}^{1} y\sqrt{1 + y'^2}\, dx,$$

subject to the constraint

$$\int_{-1}^{1} \sqrt{1 + y'^2}\, dx = 2L,$$

where $L > 1$. Show that $y - y_0 = k \cosh((x - x_0)/k)$ for constants x_0, y_0 and k. By applying the boundary conditions and considering the symmetries of the cosh function, or otherwise, show that $x_0 = 0$. Hence show that k satisfies

$$L = k \sinh\left(\frac{1}{k}\right).$$

6. (a) Suppose that $f = f(x_1, \ldots, x_n)$ satisfies

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^2 f(x_1, \ldots, x_n)$$

for all real numbers λ. Show that

$$\sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j} = 2f.$$

(b) The Euler-Lagrange equations for the functional $F = F(x, y_1, \ldots, y_n, y_1', \ldots, y_n')$, where $y_1, \ldots, y_n$ are dependent variables, are

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y_j'}\right) - \frac{\partial F}{\partial y_j} = 0,$$

where $j = 1, \ldots n$. Show that

$$\frac{d}{dx}\left(F - \sum_{j=1}^{n} y_j' \frac{\partial F}{\partial y_j'}\right) - \frac{\partial F}{\partial x} = 0.$$

Let $L = T - V$ be the Lagrangian of a particle moving under the action of a conservative force. Show that if $L = L(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ satisfies

$$\frac{\partial L}{\partial t} = 0,$$

then the total energy $E = T + V$ is constant throughout the motion.

7. Let $(r(t), \theta(t))$ be polar coordinates for the position of a particle of mass m at time t acted on by a conservative force corresponding to the central potential $V = V(r)$. Define the Lagrangian of the motion and show that

i) $h = r\dot{\theta}^2$ is constant,

ii) $m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$.

Suppose that $V(r) = -\frac{\gamma m}{r}$ for some constant γ. By writing $r = r(\theta)$ and eliminating the time dependency from ii), show that

$$-\frac{1}{r^2}\frac{\partial^2 r}{\partial \theta^2} + 2\frac{1}{r^3}\frac{\partial r}{\partial \theta} + \frac{1}{r} - \frac{\gamma}{h^2} = 0.$$

Hence deduce that if $u = \frac{1}{r}$, then

$$\frac{\partial^2 u}{\partial \theta^2} + u = \frac{\gamma}{h^2}.$$

Show that

$$r = \frac{h^2/\gamma}{1 + e\cos(\theta - \varphi)},$$

where e and φ are constants.