UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

M.Sc. *PG Dip*

Mathematics for General Relativity

COURSE CODE : **MATHG305**

DATE : **19-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-N0125-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Geodesic Equation ($U^a = \mathrm{d}X^a/\mathrm{d}\tau$)

$$\frac{\mathrm{d}U^a}{\mathrm{d}\tau} + \Gamma^a{}_{bc} U^b U^c = 0; \qquad (1)$$

$$\frac{\mathrm{d}p_a}{\mathrm{d}\tau} = \frac{m}{2}\left(\partial_a g_{bc}\right) U^b U^c; \qquad (2)$$

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right). \qquad (3)$$

Geodesics parameterized by s

$$\frac{\mathrm{d}^2 X^a}{\mathrm{d}s^2} + \Gamma^a{}_{bc} \frac{\mathrm{d}X^b}{\mathrm{d}s}\frac{\mathrm{d}X^c}{\mathrm{d}s} = 0. \qquad (4)$$

Schwarzschild metric line element

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2); \qquad r_s = 2GM.$$

Faraday tensor

$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}; \qquad F^{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Dual Faraday tensor

$$^*F^{ab} \equiv \frac{1}{2}\epsilon^{abcd} F_{cd} = \begin{pmatrix} 0 & +B_x & +B_y & +B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$

Maxwell Source Equations

$$\partial_b F^{ab} = j_e^a.$$

Internal Maxwell Equations

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \qquad \text{or} \qquad \partial_b {}^*F^{ab} = 0.$$

1. (a) Determine which of the following equations is a valid tensor equation. Describe the errors in the other equations.

   (i) $f = G_a{}^b K_a{}^d H^a{}_c L^a{}_d$

   (ii) $P_a = A_a{}^b B_b + U_c V^d W_d$

   (iii) $X_{ab} = Q^c{}_{bca} + U_b W_a$

   (iv) $h = \partial^a V^a - \partial^b \partial_c Z_b{}^c$

   (b) Describe how the Riemann tensor $R^a{}_{bcd}$ transforms from unprimed coordinates X to primed coordinates X'.

   (c) Consider a two-dimensional manifold M with coordinates X^1 and X^2. Suppose the metric is

$$g_{ab} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

   Also let V^a be a vector and Q_{ab}, T^{bc} be tensors with values

$$\begin{aligned} V^1 &= 4, \quad V^2 = 1; \\ Q_{11} &= 1, \quad Q_{12} = 3, \qquad Q_{21} = 5, \quad S_{22} = 7, \\ T^{11} &= 0, \quad T^{12} = 2, \qquad T^{21} = 4, \quad T^{22} = 6. \end{aligned}$$

   Find the following:

   (i) $\overline{\mathbf{V}} \cdot \overline{\mathbf{V}}$;

   (ii) $R^c{}_a = T^{bc} Q_{ba}$.

   (iii) $T^a{}_a$.

   (d) Using index notation prove the vector identity

$$\nabla \cdot (\overline{\mathbf{A}} \times \overline{\mathbf{B}}) = \overline{\mathbf{B}} \cdot \nabla \times \overline{\mathbf{A}} - \overline{\mathbf{A}} \cdot \nabla \times \overline{\mathbf{B}}.$$

   (e) Consider a four-dimensional manifold with an anti-symmetric tensor A^{ab} and a symmetric tensor S^{ab}. How many independent components does A^{ab} have? How many independent components does S^{ab} have?

   (f) Suppose a tensor A has components A_E^{ab} in the Earth frame E and components A_S^{cd} in the space frame S. Show that if A_E^{ab} is anti-symmetric then A_S^{cd} will also be antisymmetric.

2. For this question, assume Special Relativity holds (i.e. flat space with $g_{ab} = \eta_{ab}$).

   (a) Write down the Internal Maxwell Equation for $a = 0$, $b = 2$, $c = 3$, expressing this equation in terms of the electric and magnetic fields.

   (b) Show that the Internal Maxwell Equations are trivial (i.e., they give no information) if two of the indices are equal (e.g. if $a = b = 1$).

   (c) Let ϵ^{abcd} be the totally antisymmetric Levi-Civita tensor. In terms of this tensor, the dual Faraday tensor is defined as ${}^*F^{ab} \equiv 1/2\, \epsilon^{abcd} F_{cd}$. Derive a simple expression for $\partial_b {}^*F^{ab}$ from this definition and the Internal Maxwell Equations, showing your work.

   (d) Express ${}^*F^{ab} F_{ab}$ in terms of $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$ (you may use the expressions for F_{ab} and ${}^*F^{ab}$ given on the first page).

   (e) In terms of the 4-vector potential ϕ_a, the Faraday Tensor F_{ab} is $F_{ab} = \partial_b \phi_a - \partial_a \phi_b$. The helicity four-vector h^a is defined as

$$h^a = {}^*F^{ab} \phi_b.$$

   Show that h^a is conserved ($\partial_a h^a = 0$) if the electric and magnetic fields are everywhere perpendicular.

3. (a) Consider a particle of mass m moving in a geodesic around an object of mass M whose gravitational field is described by the Schwarzschild metric. The particle moves in the plane $\theta = \pi/2$. The Schwarzschild metric in this plane has two symmetries. What are they? Show from the goedesic equation or Noether's theorem that the particle's orbit has two conserved quantities k and h. Express $\mathrm{d}t/\mathrm{d}\tau$ and $\mathrm{d}\phi/\mathrm{d}\tau$ in terms of k and h.

   (b) Change variables from r to $u = 1/r$ and parametrize the orbit by ϕ rather than τ. Find the orbit equation for $(\mathrm{d}u/\mathrm{d}\phi)^2$.

   (c) Let $E = mk$ and $L = mh$. From the orbit equation, derive the second order equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} = \frac{r_s m^2}{2L^2} - u + \frac{3}{2} r_s u^2.$$

   (d) Consider a photon with $m = 0$. Show that the photon has a circular orbit at

$$u = u_c = \left(\frac{2}{3r_s}\right).$$

   Next suppose that a photon is in a nearby orbit, with $u = u_c(1+\epsilon)$ with initial condition $\mathrm{d}\epsilon/\mathrm{d}\phi(\phi = 0) = 0$. What is the differential equation for ϵ? Show that the photon will either spiral in to $r = 0$ or escape to $r = \infty$.

4. Consider a two-dimensional surface embedded in three-dimensional Euclidean space (e.g. the surface of a bowl). Using cylindrical coordinates (r, ϕ, z), the surface is specified by the function $z = 2\sqrt{r-1}$ for $r > 1$.

(a) Letting $\mathsf{X}^1 = r$ and $\mathsf{X}^2 = \phi$, show that the metric of this surface is

$$g_{ab} = \begin{pmatrix} \frac{r}{r-1} & 0 \\ 0 & r^2 \end{pmatrix} .$$

Also find g^{bc}.

(b) Calculate the Christoffel symbols $\Gamma^a{}_{bc}$ for this metric.

(c) From the geodesic equation, find the differential equations for a geodesic, i.e. find $\mathrm{d}^2 r/\mathrm{d}s^2$ and $\mathrm{d}^2\phi/\mathrm{d}s^2$ for a geodesic parameterized by s.

(d) Show that the metric has a symmetry, and hence there exists a conserved quantity (call it K) along each geodesic. Express $\mathrm{d}\phi/\mathrm{d}s$ in terms of K.

5. Consider the surface of the Earth (assumed to be perfectly spherical). In terms of its radius R, co-latitude θ and longitude ϕ, the metric line element is

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

(a) Suppose a ship at some position (θ, ϕ) travels with compass bearing ψ (e.g. $\psi = 0$ if the ship is heading North, and $\psi = \pi/4$ if the ship is heading Northeast). If the ship travels a small distance, with a change $\delta\theta$ in co-latitude and a change $\delta\phi$ in longitude, what is the ratio $\delta\theta/\delta\phi$ in terms of ψ?

(b) A Mercator map projection uses coordinates (x, y), where the coordinate transformations are

$$\begin{aligned} x &= a\,\phi, \\ y &= a \log \cot \frac{\theta}{2} \end{aligned}$$

with a constant. Find $\delta y/\delta x$ in terms of $\delta\theta/\delta\phi$. Consider a direction on the Mercator map which makes an angle of $\tilde{\psi}$ with respect to the vertical. Show that $\tilde{\psi} = \psi$.

(c) Find the metric line element ds^2 for the Mercator coordinates x and y (hint: you may use the identity $\sin\theta = 1/\cosh(\log\cot\frac{\theta}{2})$ without proof).

(d) A *conformally flat* metric in two dimensions has the form

$$g_{ab} = f \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where f is some function of position. What is the function f for the Mercator map? An angle η between two vectors V^a and W^b can be defined by

$$\cos^2\eta \equiv \frac{(g_{ab}V^aW^b)^2}{(g_{cd}V^cV^d)\,(g_{ef}W^eW^f)}.$$

Show that the angle η is independent of the function f.

(e) A polar map projection uses coordinates $(\mathrm{X}^1, \mathrm{X}^2) = (\rho, \lambda)$ where

$$\rho = R\sin\theta, \lambda = \phi.$$

What is the metric in these coordinates? Is this metric conformally flat?