UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sci.

Mathematics M311: Introduction to Second-order Elliptic Partial Differential Equations

COURSE CODE	:	**MATHM311**
UNIT VALUE	:	**0.50**
DATE	:	**09-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1055-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let $\Phi : \mathbb{R}^3 \to \overline{\mathbb{R}}$ be defined by

$$\Phi(x) = \begin{cases} \frac{1}{\|x\|} & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

Find the distributional Laplacian of Φ.

Show that the indicator function u of the set $\{(x,y) \in \mathbb{R}^2 : x \geq y\}$ is a solution of the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0,$$

where the derivatives are in the sense of distributions.

2. Define the notion of superharmonic and subharmonic functions.

Let Ω be an open subset of $\mathbb{R}^n$. Suppose that $u : \Omega \to (-\infty, \infty]$ is lower semi-continuous and has the property that for every $x \in \Omega$ there is $r_x > 0$ such that for every $0 < r < r_x$,

$$u(x) \geq \fint_{\partial B(x,r)} u(y)\, dH^{n-1}(y).$$

Show that u is superharmonic in Ω.

Show that the function Φ from Question 1 is superharmonic on $\mathbb{R}^3$.

3. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $g \in C(\partial\Omega)$. Describe Perron's definition of a generalized solution of the Dirichlet problem

$$\begin{cases} \Delta u &= 0 & \text{in} & \Omega \\ u &= g & \text{on} & \partial\Omega \end{cases},$$

stating without proof all results required for the definition.

Prove Wiener's theorem on the coincidence of upper and lower Perron's solutions.

Find a continuous function g defined on the boundary of $\Omega = \{x \in \mathbb{R}^3 : 0 < \|x\| < 1\}$ for which the Dirichlet problem has no classical solution and find Perron's solution of this problem.

4. State and prove the Lax-Milgram theorem.

Let L be an elliptic second order partial differential operator in divergence form. Explain the notion of weak solution of the boundary value problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

and give, with reasons, conditions under which the Lax-Milgram theorem may be used to show the existence of a solution.

5. State and prove Morrey's inequality.

Explain how Morrey's inequality together with the Gagliardo-Nirenberg-Sobolev inequality (which should be stated) may be used to show the interior regularity of solutions of elliptic partial differential equations.

END OF PAPER