UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sci.

Mathematics M311: Introduction to Second-order Elliptic Partial Differential Equations

COURSE CODE	:	**MATHM311**
UNIT VALUE	:	**0.50**
DATE	:	**21-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0969-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State and prove the weak and strong maximum principles for harmonic functions on open subsets of $\mathbb{R}^n$.

   (b) Show that the strong maximum principle for harmonic functions on U fails whenever U is a disconnected open subset of $\mathbb{R}^n$.

2. (a) For $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ define the notion of the weak partial derivative u_{x_i} and define the Sobolev spaces $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$. Show that the function

$$u(x) = \frac{1}{\log |x|}$$

   where $|x| = \sqrt{\sum_{i=1}^n x_i^2}$ belongs to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if $p \leq n$.

   (b) Show that for any $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ there is a sequence $u^j \in C^\infty_c(\mathbb{R}^n)$ such that $\lim_{j \to \infty} \|u - u_j\|_{W^{1,p}(V)} = 0$ for any bounded open $V \subset \mathbb{R}^n$.

3. (a) State and prove Morrey's inequality.

   (b) Determine all p for which every function $u \in W^{1,p}(\mathbb{R}^3)$ is equivalent to a continuous function. (The results from Question 2 may be used even if you did not show them.)

4. In this question we consider the boundary value problem

$$\begin{cases} Lu + \mu u = f & \text{in} \quad U \\ u = 0 & \text{on} \quad \partial U \end{cases} \tag{$\star$}$$

where $U \subset \mathbb{R}^n$ is bounded and open, $\mu \in \mathbb{R}$, $f \in L^2(U)$ and L is a (uniformly) elliptic second order partial differential operator in divergence form, so

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u, \text{ with } a^{ij}, b^i, c \in L^\infty(U).$$

(a) Explain what is meant by saying that L is (uniformly) elliptic and that u is a weak solution of ($\star$). Show that for sufficiently large μ the problem ($\star$) has a unique weak solution.

(b) In the special case when $b^i = 0$ and $c \geq 0$ show that the problem ($\star$) has a unique weak solution for every $\mu \geq 0$.

5. Let

$$Lu = -\sum_{i,j=1}^{n} \left(a^{i,j}u_{x_i}\right)_{x_j}$$

be a symmetric uniformly elliptic operator on a bounded connected open set $U \subset \mathbb{R}^n$ such that $a^{i,j} \in C^\infty(\overline{U})$.

(a) State the basic properties of eigenvalues and eigenvectors of L. What is meant by the principal eigenvalue? Also state the variational formula for the principal eigenvalue.

(b) Show that the principal eigenvalue is simple.