UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M311: Introduction to Second-order Elliptic Partial Differential Equations

COURSE CODE	:	**MATHM311**
UNIT VALUE	:	**0.50**
DATE	:	**08-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0961-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Suppose that $u \in L^1_{\text{loc}}(\mathbb{R}^n)$. Define the notion of the weak partial derivative u_{x_i} of u and show that if $u, v \in L^1(\mathbb{R})$, u_{x_i} exists and $u_{x_i} \in L^1(\mathbb{R})$ then $(u*v)_{x_i} = u_{x_i} * v$.

   (b) Find the weak partial derivative u_{x_1} of the function $u(x_1, x_2) = \sqrt{|x_1 x_2|}$.

2. (a) Give the notion of the mean value property of a locally integrable function in an open set U and show that weak solutions of Laplace's equation necessarily have this property.

   (b) Show that there is no continuous function u on

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$$

   which is harmonic on

$$U := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1^2 + x_2^2 + x_3^2 < 1\}$$

   and satisfies $u(x) = |x|$ whenever $x \in \partial U$.

3. (a) State the Gagliardo-Nirenberg-Sobolev inequality and prove it for $p = 1$.

   (b) Use the Gagliardo-Nirenberg-Sobolev inequality and Morrey's inequality to find out for which k it is true that every function from $H^k(\mathbb{R}^3)$ is equivalent to a continuous function.

4. (a) State and prove the Lax-Milgram Theorem.

   (b) Let $U = B_{0,1}$ be the unit disk in the plane and consider the partial differential operator $Lu = -u_{xx} - u_{yy} - (gu_x)_y$, where $g(x, y) = 1$ if $x \geq y$ and $g(x, y) = 0$ otherwise. Use the Lax-Milgram theorem to show that the boundary value problem

$$\begin{cases} Lu + u &= 1 \quad \text{in} \quad U \\ u &= 0 \quad \text{on} \quad \partial U \end{cases}$$

   has a weak solution in $H^1_0(U)$.

5. (a) Suppose that Lu is an elliptic partial differential operator (in divergence form) on an open subset U of $\mathbb{R}^n$ with coefficients belonging to $C^{m+1}(U)$ and that $f \in H^m_{\text{loc}}(U)$. State a result giving the maximal regularity (in the sense of belonging to $H^k_{\text{loc}}(U)$ spaces) of any weak solution of the equation

$$Lu = f \qquad \text{in } U$$

and assuming that it holds for $m = 0$ prove it for all other m.

(b) Use the result from (a) to find the largest k for which every weak solution u of

$$-\Delta u + ((2 + \cos(xyz))^{3/2} u_x)_y - (|\sin(xyz)|^{5/2} u_y)_z = 1 \qquad \text{in } \mathbb{R}^3$$

belongs to $H^k_0(U)$.