UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C329: Functions Of A Complex Variable I

COURSE CODE : **MATHC329**

UNIT VALUE : **0.50**

DATE : **27-APR-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-C1009-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the maximal function $M(r)$.

   (b) State and prove the maximum principle.

   (c) Let $z_1, z_2, \ldots, z_n$ be points on the unit circle in the complex plane. Use the maximum principle to prove that there exists a point z on the unit circle such that the product of the distances from z to the points z_j, $1 \leq j \leq n$, is larger than 1. Conclude that there exists a point z on the unit circle such that the product of the distances from z to the points z_j, $1 \leq j \leq n$, is exactly 1.

2. Let $\mathcal{F}$ be a family of functions defined on a domain D. Let A be an arbitrary subset of D.

   (a) Define each of the following:

      (i) $\mathcal{F}$ is uniformly bounded on A;

      (ii) $\mathcal{F}$ is equicontinuous on A;

      (iii) $\mathcal{F}$ is normal.

   (b) State Arzela-Ascoli's theorem and Montel's theorem.

   (c) Let

   $$f_n(z) = \begin{cases} 1 - nz & \text{if } |z| \leq \frac{1}{n} \\ 0 & \text{if } |z| > \frac{1}{n}. \end{cases}$$

   Show that $\mathcal{F} = \{f_1, f_2, \ldots\}$ is uniformly bounded on every compact set, but not normal. Why does this not contradict Montel's theorem?

3. (a) Show that the conformal automorphisms of the unit disc $B(0,1)$ are the mappings

   $$f(z) = c \cdot \frac{z - z_0}{z\overline{z_0} - 1},$$

   where $|c| = 1$ and $z_0 \in B(0,1)$. (You may use without proof that every conformal automorphisms of the unit disc is a fractional linear transform.)

   (b) A complex number z is a *fixed point* for a conformal map f if $f(z) = z$. Prove that if f is a conformal automorphism of the unit disc and has two distinct fixed points, then f is the identity, that is, $f(z) = z$ for all z.

4. (a) Define the order of growth of an entire function and define the canonical factors.
   (b) State Hadamard's product theorem.
   (c) Show that the equation $e^z - z = 0$ has infinitely many solutions in $\mathbb{C}$. (You may use without proof that the order of growth of $e^{p(z)}$ is $\deg p$ for every polynomial $p(z)$.)

5. (a) State the little Picard theorem and Picard's theorem.
   (b) Prove Picard's theorem. (You may use the lemma that there is a constant $c > 0$ such the image of every holomorphic function f on the unit disc $B(0, 1)$ with $f(0) = 0$, $f'(0) = 1$ contains a disc of radius c.)