UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M231: Fluid Mechanics

COURSE CODE	:	**MATHM231**
UNIT VALUE	:	**0.50**
DATE	:	**07-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0947-3-90

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

The fluid is incompressible and inviscid and, except where noted in Question 4, has constant density ρ. Gravitational acceleration is denoted by g throughout.

1. (a) Define the terms streamline, particle path and streakline. A fluid moves two-dimensionally so that its velocity $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = (\cosh t)\boldsymbol{i} + (\sinh t)\boldsymbol{j},$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the unit vectors for the Cartesian coordinates (x, y).
Obtain equations in terms of x and y alone for the following:

(i) the streamline through (0,1) at time $t = 0$,
(ii) the particle path for a particle released from (0,1) at time $t = 0$,
(iii) the streakline through (0,1) formed by particles released from (0,1) at times $t \leqslant 0$.

Sketch the three loci on the same diagram in the (x, y) plane.

(b) The volume flux F of fluid crossing a closed surface S is defined as

$$F = \int_S \boldsymbol{n} \cdot \boldsymbol{u} \, \mathrm{d}S,$$

where $\boldsymbol{u}$ is the fluid velocity and $\boldsymbol{n}$ is the normal to the surface S.
By considering an arbitrary sub-region, or otherwise, show that if there are no sources or sinks of fluid within a domain D then $\boldsymbol{u}$ satisfies

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{within} \quad D.$$

2. A stationary circular cylinder with boundary $x^2 + y^2 = a^2$ is in a two-dimensional *rotational* flow whose velocity has Cartesian components (u, v) such that $u - ky \to 0$ and $v \to 0$ as $(x^2 + y^2)/a^2 \to \infty$, where k is a positive constant. The circulation around the cylinder is $-\pi k a^2$.

   (a) State with brief reasons why it may be preferable to use a streamfunction rather than a velocity potential to express the flow field.

   (b) Give a brief argument to show that the partial differential equation satisfied by a streamfunction ψ can be written

   $$\nabla^2 \psi = k.$$

   (c) Show that the streamfunction in the far-field can be written

   $$\psi_{ff} = \frac{1}{2} k y^2.$$

   (d) By writing

   $$\psi = \psi_{ff} + \psi_0,$$

   show that ψ_0 is irrotational and has zero circulation about the cylinder. What boundary conditions does ψ_0 satisfy?

   (e) Hence, or otherwise, find ψ_0 and so ψ.

3. A uniform stream of clear fluid with speed U at infinity flows past a circular cylinder of radius a. The surface of the cylinder is porous and dyed fluid is forced out with outward normal velocity $2U$ at the surface.

   (a) By finding the velocity at large distance and on $|z| = a$, show that the flow field can be represented by the complex velocity potential

   $$F(z) = U(z + a^2/z) + 2aU \log z,$$

   where $z = 0$ is at the centre of the circular cross-section of the cylinder.

   You may find the following relation between the complex velocity potential and the polar components of the velocity useful:

   $$e^{i\theta} \frac{dw}{dz} = u_r - i u_\theta.$$

   (b) Show that the dyed fluid extends a distance $(1+\sqrt{2})a$ upstream from the centre of this circle and that far downstream it occupies a region of width $4\pi a$.

   (c) Sketch the streamlines of the motion, distinguishing between the dyed and undyed fluid.

4. The Reynolds' transport theorem for a scalar quantity α can be written

$$\frac{D}{Dt}\int_{\mathcal{V}} \alpha \; \mathrm{d}V = \int_{\mathcal{V}} \frac{\partial \alpha}{\partial t} \; \mathrm{d}V + \int_{S} \alpha \boldsymbol{u} \cdot \boldsymbol{n} \; \mathrm{d}S.$$

(a) Explain briefly the meaning of each term in this expression.

(b) By considering the conservation of mass in a fluid of variable density $\rho(x, y, z, t)$, use the Reynolds' transport theorem to derive

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0.$$

(c) Using this result show that the transport theorem for a scalar quantity β can be written

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho\beta \; \mathrm{d}V = \int_{\mathcal{V}} \rho\frac{D\beta}{Dt} \; \mathrm{d}V$$

(d) By considering the momentum balance in the presence of external forces $\mathbf{F}$ per unit mass, obtain the Euler equations

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho\mathbf{F}.$$

(e) Using Euler's equations in Cartesian form or otherwise show that under gravitational force alone the free surface of a fluid of constant density in solid body rotation with

$$\boldsymbol{u} = \Omega(-y\boldsymbol{i} + x\boldsymbol{j}),$$

is a paraboloid.

You may use without proof the results

$$\int_{S} p\boldsymbol{n} \; \mathrm{d}S = \int_{\mathcal{V}} \nabla p \; \mathrm{d}V$$

and

$$\nabla \cdot (\phi \boldsymbol{u}) = \phi \nabla \cdot \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\phi.$$

5. A small-amplitude wave is progressing in the positive x-direction on the surface of water of density ρ and mean depth h, so that the equation of the surface is $y = \eta(x, t)$ where y is measured vertically upwards from the undisturbed surface. The linearised dynamic boundary condition at the surface, $y = 0$, can be written

$$\frac{\partial \phi}{\partial t} = -g\eta,$$

where ϕ is the velocity potential and g is the acceleration due to gravity.

(a) By considering the motion of a particle on the free surface show that the other linearised boundary condition on ϕ at $y = 0$ is

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}.$$

(b) State, and briefly justify, the boundary condition on ϕ at $y = -h$.

(c) State the partial differential equation satisfied by ϕ.

(d) If $\eta(x, t) = \epsilon \sin[(2\pi/\lambda)(x - ct)]$ with $\epsilon \ll \lambda$, $\epsilon \ll h$, show that the velocity potential is

$$\phi = -\epsilon c \frac{\cosh[(2\pi/\lambda)(y + h)]}{\sinh(2\pi h/\lambda)} \cos[(2\pi/\lambda)(x - ct)],$$

where the wavespeed is given by the dispersion relation

$$\frac{c^2}{gh} = \frac{\lambda}{2\pi h} \tanh\left(\frac{2\pi h}{\lambda}\right).$$

(e) A second wave of the same amplitude (but different phase) propagates in the opposite direction so that its surface displacement is given by

$$\eta_2 = -\epsilon \sin[(2\pi/\lambda)(x + ct)].$$

Use the above results to write down the velocity potential associated with this wave and the velocity potential associated with the combined surface elevation $\eta(x, t) + \eta_2(x, t)$. Deduce that the combined potential can represent the flow when a progressive surface wave reflects at a solid wall at $x = 0$.

You may use without proof the identity

$$\cos A + \cos B = 2 \cos\left[\frac{1}{2}(A + B)\right] \cos\left[\frac{1}{2}(A - B)\right].$$

6. Consider a shallow stream flowing along a constant-width channel. Let the channel floor rise slowly by a small amount $k > 0$. Let the fluid depth upstream of the rise be h_1, the fluid depth downstream of the rise be h_2, and the rise of the surface downstream (relative to the surface upstream) be r.

   (a) Use Bernoulli's equation to show that

$$H(h_1) = H(h_2) + k,$$

   where $H(h) = h + Q^2/(2gh^2)$ and Q is a constant, proportional to the mass flux along the channel.

   (b) Sketch the graph of $H(h)$ obtaining the position h_m of the minimum of H (for $h > 0$).

   (c) Show that

$$r = \frac{Q^2}{2g}(h_1^{-2} - h_2^{-2}).$$

   (d) Describe the difference in the behaviour of the flow depending on whether h_1 is greater or less than h_m.