UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C371: Analytic Theory Of Numbers

COURSE CODE	:	**MATHC371**
UNIT VALUE	:	**0.50**
DATE	:	**23-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1019-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let $a(n)$ be an arithmetic function, and let $A(x)$ be its summation function $A(x) = \sum_{n \leq x} a(n)$. Let f be a continuously differentiable function on the interval $[y, x]$. Express the sum

$$\sum_{y<n\leq x} a(n)f(n)$$

using the continuous version of Abel summation.

(b) Use Abel summation to show that

$$\sum_{p} \frac{1}{p} = \infty \quad \text{and} \quad \sum_{p} \frac{1}{p \log p} < \infty$$

where the summation is taken over all primes. (Prime number theorem can be used without proof.)

2. (a) Define the following notions:

(i) an arithmetic function is multiplicative

(ii) an arithmetic function is completely multiplicative

(iii) convolution of two arithmetic functions.

(b) Let $\sigma(n) = \sum_{j|n} j$ and let $S(x) = \sum_{n \leq x} \sigma(n)$.

(i) Show that $\sigma = \mathbf{1} * \mathrm{id}$, where $\mathbf{1}(n) = 1$ and $\mathrm{id}(n) = n$ for all n.

(ii) Deduce that $S(x) = \frac{\pi^2}{12}x^2 + q(x)$, where $|q(x)| = O(x \log x)$.

3. (a) Let $a(n) = \mu(n) \log n$, where $\mu(n)$ denotes the Möbius function, and let $A(x) = \sum_{n \le x} a(n)$. Which function has the Dirichlet series $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$? Use the integral version of the prime number theorem to show that

$$\int_1^{\infty} \frac{A(x)}{x^2}\, dx = -1.$$

You may use any valid formula for $\zeta(s)$ and $\zeta'(s)$ without proof.

(b) Prove the main lemma of Newmann's proof of the integral version of prime number theorem, namely: if $B : \mathbb{R} \to \mathbb{R}$ is continuous except possibly at the integers, and has left and right limits at each integer, and

- $B(t) = O(1/t)$ as $t \to \infty$;
- $\int_1^{\infty} \frac{B(t)}{t^s}\, dt = g(s)$ is a holomorphic function on the halfplane $\{\text{Re}\, s > 0\}$;
- $g(s)$ has a holomorphic extension to a domain that contains $\{\text{Re}\, s \ge 0\}$;
- $g(0) = 0$

then $\int_1^{\infty} B(t)\, dt$ converges and equals to 0. (The Riemann-Lebesgue Lemma can be used without proof.)

4. (a) Define what is

(i) a character of a finite Abelian group

(ii) a Dirichlet character mod k

(iii) the function $L(\chi, s)$.

(b) Let χ be any Dirichlet character mod k, and let χ_0 be the principal character mod k. Show the following steps of the proof of Dirichlet's theorem about primes in residue classes.

(i) Write the Euler product for χ_0 and deduce that it has a pole at $s = 1$ with residuum $\frac{\phi(k)}{k}$.

(ii) Show that for any $\sigma > 1$ and for any t,

$$\left| L(\chi_0, \sigma)^3 L(\chi, \sigma + it)^4 L(\chi^2, \sigma + 2it) \right| \ge 1.$$

You may use without proof any valid formula for $L(\chi, s)$ and $\log L(\chi, s)$.

(iii) Deduce that if $\chi^2 \ne \chi_0$ or $t \ne 0$, then $L(\chi, 1 + it) \ne 0$.

5. (a) Let $c_k(n)$ be the number of k-tuples of primes $(p_1, p_2, \ldots, p_k)$ for which $n = p_1 p_2 \cdots p_k$. Find an arithmetic function $a(n)$ for which c_k is the k-fold convolution $c_k = a * a * \cdots * a$.

(b) Let

- $\Pi_k(x)$ = number of positive integers $n \leq x$ that are the product of k distinct primes;
- $\pi_k(x)$ = number of positive integers $n \leq x$ that are the product of k arbitrary primes;
- $N_k(x)$ = number of k-tuples of primes $(p_1, \ldots, p_k)$ whith $p_1 p_2 \cdots p_k \leq x$;
- $$L_k(x) = \sum_{p_1 p_2 \cdots p_k \leq x} \frac{1}{p_1 p_2 \cdots p_k};$$
- $$\Theta_k(x) = \sum_{p_1 p_2 \cdots p_k \leq x} \log(p_1 p_2 \cdots p_k).$$

Show the following three main steps in the proof of

$$\Pi_k(x) \sim \pi_k(x) \sim \frac{x}{\log x} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!},$$

(you may assume their validity for $k = 1$ without proof):

(i) Show that $L_k(x) \sim (\log \log x)^k$.

(ii) Deduce that $\Theta_k(x) \sim kx(\log \log x)^{k-1}$. You may use without proof the recursive formula

$$F_k(x) = \frac{k}{k-1} \sum_{p \leq x} F_{k-1}\left(\frac{x}{p}\right)$$

for $F_k(x) = \Theta_k(x) - kxL_{k-1}(x)$.

(iii) Deduce that $N_k(x) \sim \frac{x}{\log x} \cdot k(\log \log x)^{k-1}$.