UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C371: Analytic Theory Of Numbers

COURSE CODE	:	**MATHC371**
UNIT VALUE	:	**0.50**
DATE	:	**12-MAY-05**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

05-C0936-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define what it means that
   (i) an arithmetic function is multiplicative
   (ii) an arithmetic function is completely multiplicative
   (iii) convolution of two arithmetic functions.

   (b) Let $\sigma_r(n) = \sum_{j|n} j^r$.
   (i) Find an arithmetic function $a(n)$ such that $\sigma_r = a * \mathbf{1}$.
   (ii) Express $\sum_{n=1}^{\infty} \frac{\sigma_r(n)}{n^s}$ in terms of the ζ-function for suitable s.
   (iii) Deduce that $\sigma_{-1}(n) = \frac{\sigma_1(n)}{n}$.

2. Suppose that $a(n)$ is completely multiplicative and $S_1 = \sum_{n=1}^{\infty} a(n)$ is absolutely convergent.

   (a) State Euler's product formula and state the corresponding formula for the ζ-function.

   (b) Define the Möbius function μ and use Euler's product formula to show that $\frac{1}{S_1} = \sum_{n=1}^{\infty} \mu(n)a(n)$.

   (c) Let $S_2 = \sum_{n=1}^{\infty} a(n)^2$. Show that $\frac{S_1}{S_2} = \sum_{n=1}^{\infty} |\mu(n)|a(n)$ and state the corresponding formula for ζ-function. (Hint: use the equality $\frac{1-a(p)^2}{1-a(p)} = 1 + a(p)$.)

3. Show the main step of the proof of prime number theorem, namely: if f is a holomorphic function on a domain that contains $\{\operatorname{Re} s \geq 1\}$ except possibly for $s = 1$, and $a(n)$ is an arithmetic function with $A(x) = \sum_{n \leq x} a(n)$ such that

- $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ converges absolutely to $f(s)$ when $\operatorname{Re} s > 1$;
- $f(s) = \frac{\alpha}{s-1} + \beta + (s-1)h(s)$, h is holomorphic on a domain that contains $\{\operatorname{Re} s \geq 1\}$;
- there is a function $P(t)$ such that $|f(\sigma \pm it)| \leq P(t)$ when $\sigma \geq 1$ and $t \geq t_0$ ($t_0 \geq 1$), and also $\int_1^{\infty} \frac{P(t)}{t^2}\, dt < \infty$,

then

$$\int_1^{\infty} \frac{A(y) - \alpha y}{y^2}\, dy = \alpha - \beta.$$

You may use without proof the Riemann-Lebesgue Lemma, and the facts that $\left|\frac{h(s)}{s}\right| \leq \frac{P(t)+|\alpha|+|\beta-\alpha|}{t^2}$ (for $s = \sigma \pm it$, $\sigma \geq 1$, $t \geq t_0$) and $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}h(s)}{s}\, ds = \int_1^x \frac{A(y)-\alpha y}{y^2}\, dy - (\beta - \alpha)\left(1 - \frac{1}{x}\right)$ (for $x \geq 1$, $c > 1$).

4. (a) Define what is
   (i) a character of a finite Abelian group
   (ii) a Dirichlet character mod k
   (iii) the function $L(\chi, s)$.

   (b) Let χ be any Dirichlet character mod k, and let χ_0 be the principal character mod k. Show that for any $\sigma > 1$ and for any t,

$$\left|L(\chi_0, \sigma)^3 L(\chi, \sigma + it)^4 L(\chi^2, \sigma + 2it)\right| \geq 1.$$

   You may use without proof any valid formula for $L(\chi, s)$ and $\log L(\chi, s)$.

   (c) Explain where $L(\chi, s) \neq 0$ was used in the proof of Dirichlet's theorem about prime numbers in residue classes.

5. (a) Define the function $\mathrm{li}(x)$ and state prime number theorem together with error estimate.

   (b) Show that

$$I_n(x) \stackrel{\text{def}}{=} \int_e^x \frac{1}{(\log t)^n}\, dt \sim \frac{x}{(\log x)^{n+1}}.$$

   (c) Deduce that

$$|\pi(x) - \mathrm{li}(x)| < \left|\pi(x) - \frac{x}{\log x}\right|$$

   for every large enough x, i.e. $\mathrm{li}(x)$ is a better approximation of $\pi(x)$ than $\frac{x}{\log x}$.