UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C371: Analytic Theory Of Numbers

COURSE CODE	:	**MATHC371**
UNIT VALUE	:	**0.50**
DATE	:	**04-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0931-3-30

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

Throughout the paper p denotes a prime number, $d|\,n$ means "d divides n" and the complex variable s equals $\sigma + it$. Also $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ denotes the Riemann zeta-function.

1. Define the terms "multiplicative" and "completely multiplicative" for an arithmetic function $\alpha(n)$. Define the totient function $\phi(n)$ and the von Mangoldt function $\Lambda(n)$ and determine whether or not they are completely multiplicative.

   Show that $\sum_{d|n} \phi(d) = n$ for $n \in \mathbb{N}$.

2. (i) What is meant by a Dirichlet Character (mod k) and a principal character (mod k)?

   (ii) If $\mathcal{X}$ is a non-principal character (mod k) show that, under suitable conditions on the function $f(x)$, to be clearly stated,

   $$\sum_{n>x} \mathcal{X}(n) f(n) = O(f(x)) \qquad (x \to \infty),$$

   and deduce that $L(1, \mathcal{X}) = \sum_{n=1}^{\infty} \frac{\mathcal{X}(n)}{n}$ converges for non-principal $\mathcal{X}$.

   State Dirichlet's theorem on primes in arithmetic progressions and explain (without detailed proofs) why it is important to know that $L(1, \mathcal{X}) \neq 0$ for a non-principal character $\mathcal{X}$?

3. State the prime number theorem and define the Chebyshev functions $\psi(x)$ and $\theta(x)$. If $\pi(x)$ denotes the number of prime numbers $\leqslant x$ show that, for $x \geqslant 2$,

   $$\theta(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt$$

   $$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2} dt.$$

   Show that the prime number theorem is equivalent to the statement that $\lim_{x \to \infty} \frac{\theta(x)}{x} = 1$.

4. Define the Chebyshev function $\psi_1(x)$ and show that

$$\psi_1(x) = \sum_{n \leqslant x} (x-n)\Lambda(n).$$

Hence show that if $c > 1$

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)} \left[-\frac{\zeta'(s)}{\zeta(s)} \right] ds.$$

The relation

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{u^{-z}}{z(z+1)\ldots(z+k)} dz = \begin{cases} \frac{1}{k!}(1-u)^k & 0 < u \leqslant 1 \\ 0 & u > 1 \end{cases}$$

if $c > 0$, may be assumed.

5. Define the Hurwitz ζ-function $\zeta(s,a)$, the Hurwitz L-function $F(x,s)$ and the Gamma function $\Gamma(s)$.

For which values of σ are these definitions valid?

By considering

$$I_N(s,a) = \frac{1}{2\pi i} \int_{C_N} \frac{z^{-s} e^{az}}{1-e^z} dz$$

where C_N is the circle $|z| = 2N+1$ with a slit on the negative real axis from $-(2N+1)$ to 0 prove that

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s} \left\{ e^{-\frac{\pi i s}{2}} F(a,s) + e^{\frac{\pi i s}{2}} F(-a,s) \right\}.$$

Problems of convergence arising as $N \to \infty$ should be clearly stated but proofs of the estimates required need not be given.