UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C371: Analytic Theory Of Numbers

COURSE CODE	:	**MATHC371**
UNIT VALUE	:	**0.50**
DATE	:	**16-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0925-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Throughout the paper p denotes a prime number, d/n means "d divides n" and the complex variable s equals $\sigma + it$. Also $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ denotes the Riemann Zeta-Function.

1. Define the Dirichlet Multiplication of two arithmetic functions $a(n)$ and $b(n)$ denoted by $a * b$. Show that, with respect to the operation $(*)$ the set of arithmetic functions with $a(1) \neq 0$ forms an Abelian group. Define the Möbius function $\mu(n)$ and the Euler Totient function $\phi(n)$. State Möbius' inversion formula and use it to show that

$$\phi(n) = \sum_{d/n} \mu(d) \frac{d}{n}.$$

You may assume that $\sum_{d/n} \phi(d) = n$.

2. If $F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, $G(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$ and $F(s)G(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$, all series assumed to be absolutely convergent for $\sigma > \sigma_0$, find a formula for $h(n)$.

   Define the abcissa of absolute convergence σ_a and the abcissa of conditional convergence σ_c of a Dirichlet series. Show that $0 \leqslant \sigma_a - \sigma_c \leqslant 1$ and give an example where $\sigma_a - \sigma_0 = 1$. Show that $\zeta(s) \left\{ \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right\} = 1$ if $\sigma > 1$ and deduce that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$.

3. State the prime number theorem. Define the von Mangoldt function $\Lambda(n)$ and the Chebyshev functions $\theta(x)$, $\psi(x)$, $\psi_1(x)$. Show that

$$\lim_{n\to\infty} \frac{\psi(x)-\theta(x)}{x} = 0.$$

Given that the prime number theorem is equivalent to the statement that $\lim_{n\to 0} \frac{\theta(x)}{x} = 1$ show that it is also equivalent to the statement

$$\lim_{x\to\infty} \frac{\psi_1(x)}{x^2} = \frac{1}{2}.$$

(Any theorem of a Tauberian nature used should be clearly stated but need not be proved.)

4. Define the Dirichlet characters (mod k) and write down the table of Dirichlet characters (mod 7). *State* Abel's Lemma and use it to show that for any non-principal character $\mathcal{X}$(mod k) we have, for $x \geqslant 1$

$$\sum_{n\leqslant x} \frac{\mathcal{X}(n)}{n} = \sum_{n=1}^{\infty} \frac{\mathcal{X}(n)}{n} + O\left(\frac{1}{x}\right) \quad \text{as } x \to \infty.$$

If $L(s,\mathcal{X}) = \sum_{n=1}^{\infty} \frac{\mathcal{X}(n)}{n^s}$ use Möbius' Inversion formula to show that if $\mathcal{X}$ is a non-principal character (mod k) then

$$[L(1,\mathcal{X})] \sum_{n\leqslant x} \frac{\mu(n)\mathcal{X}(n)}{n} = O(1) \text{ as } x \to \infty.$$

5. Define the Gamma function $\Gamma(s)$. For which values of s is your definition valid? What is meant by a function $f(s)$ being continued analytically outside a given region? Prove that the Riemann Zeta function $\zeta(s)$ can be continued analytically into the region $\sigma > 0$ except for $s = 1$ where there is a simple pole of residue 1.

Show that if $\sigma > 1$ then

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{x^{s-1}e^{-x}dx}{1-e^{-x}}.$$

END OF PAPER