UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

M.Sci.

Mathematics C371: Analytic Theory Of Numbers

COURSE CODE	:	**MATHC371**
UNIT VALUE	:	**0.50**
DATE	:	**01-MAY-02**
TIME	:	**10.00**
TIME ALLOWED	:	**2 hours**

02-C0921-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Define the Möbius function $\mu(n)$ and the Euler totient function $\phi(n)$ and the Liouville function $\lambda(n)$. Show that

$$\phi(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right) \text{ if } n \geqslant 1.$$

For $n \geqslant 1$ evaluate the sums

(a) $\sum_{d|n} \mu(d)$

(b) $\sum_{d|n} \lambda(d)$.

State what is meant by the terms "multiplicative" and "completely multiplicative" for an arithmetic function. Are $\mu(n)$ and $\lambda(n)$ multiplicative and completely multiplicative? Justify your answers.

2. State Abel's Lemma, indicating clearly the assumptions that you make. Define the Tchebyshev functions $\psi(x)$ and $\theta(x)$ and show that

$$0 \leqslant \frac{\psi(x)}{x} - \frac{\theta(x)}{x} \leqslant \frac{(\log x)^2}{(2\log 2)x^{\frac{1}{2}}},$$

for $x > 0$. Show that

$$\theta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)dt}{t}.$$

3. Suppose that $f(t) \geqslant 0$ for $t \geqslant 0$ and consider the two conditions

(a) $f(t) \sim t \qquad (t \to \infty)$

(b) $\int_0^x f(t)dt \sim \frac{x^2}{2} \qquad (x \to \infty)$

Show that (a) implies (b) and that (b) implies (a) if the function $g(t) = t\ f(t)$ is increasing.

State the prime number theorem. Let

$$\psi_1(x) = \int_1^x \psi(t)dt,$$

where $\psi(t)$ is the Tchebyshev function. Show that the prime number theorem implies that $\psi_1(x) \sim \frac{1}{2}x^2$ as $x \to \infty$. The results of Question 2 may be assumed.

4. Define the Dirichlet characters (mod$k$) and show that they are completely multiplicative and periodic with period k. Define the Dirichlet L-function $L(s, \chi)$. What is meant by a non-principal character? Explain why it is important for Dirichlet's theorem about primes in arithmetic progressions to know that $L(1, \chi) \neq 0$ for such a character.

5. Define the von Mangoldt function $\Lambda(n)$. If $\psi_1(x)$ is as in Question 3, show that

$$\frac{\psi_1(x)}{x} = \sum_{n \leqslant x} (x - n)\Lambda(n).$$

Show that, if $c > 1$ and $x \geqslant 1$,

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)} \left[-\frac{\zeta'(s)}{\zeta(s)} \right] ds.$$

(You may assume the relations

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{u^{-z}dz}{z(z+1)\dots(z+k)} &= \frac{1}{k!}(1-u)^k \text{ if } 0 < u \leqslant 1 \\ &= 0 \text{ if } u > 1 \end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}.)$$

Indicate clearly, but without proof, how the above representation can lead to a proof of the prime number theorem.

END OF PAPER