UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M212: Analysis 4: Real Analysis

COURSE CODE : **MATHM212**

UNIT VALUE : **0.50**

DATE : **24-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-C1044-3-100

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define what it means for a sequence of functions $\{f_n\}_{n=1}^{\infty}$ to converge *uniformly* on an interval $[a, b]$.
   (b) Prove that if $f_n \to f$ uniformly on $[a, b]$, and each f_n is continuous on $[a, b]$, then f is continuous on $[a, b]$.
   (c) Prove that if $f_n \to f$ uniformly on $[a, b]$, and each f_n is Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, b]$ and $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$ as $n \to \infty$.
   (d) Suppose that $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are two sequences of functions on $I \subset \mathbb{R}$ such that both $f_n \to f$ and $g_n \to g$ uniformly on I, where g is a positive function. Is it true that $\frac{f_n}{g_n} \to \frac{f}{g}$ uniformly on I? Justify your answer. [Hint: I does not have to be a closed interval].

2. Prove that for any continuous function f on the interval $[0, 1]$ there exists a sequence $\{B_n(f)\}_{n=1}^{\infty}$ of polynomials with $B_n(f) \to f$ uniformly on $[0, 1]$. You may use the properties of the functions p_{nk} without proof.

3. (a) Define the *Fourier series* of a Riemann integrable function f on an interval $[a, b]$ with respect to an orthonormal system $(\phi_k)_{k=1}^{\infty}$.
   (b) Let $\{a_j\}$ be the Fourier coefficients of a Riemann integrable function f on an interval $[a, b]$ with respect to an orthonormal system $(\phi_k)_{k=1}^{\infty}$ Show that for each natural n and any numbers c_1, $c_2, \ldots c_n$ the following inequality holds:
$$\int_a^b \Big(f(x) - \sum_{k=1}^{n} a_k\phi_k(x)\Big)^2 dx \le \int_a^b \Big(f(x) - \sum_{k=1}^{n} c_k\phi_k(x)\Big)^2 dx,$$
   with equality only in the case $c_j = a_j$ for all $j = 1, 2, \ldots n$.
   (c) Let $S_n = \sum_{k=1}^{n} a_k\phi_k$ be the partial sum of the Fourier series of f.
   (i) Is it true that
$$\lim_{n,m\to\infty} \int_a^b \Big(S_n(x) - S_m(x)\Big)^2 dx = 0?$$
   (ii) Is it true that S_n converges to f pointwise on $[a, b]$?
   Justify your answers.

4. (a) Define the terms metric space, open ball, closed ball, open set and closed set.
   (b) Let (X,d) be a metric space. Prove that $A \subset X$ is closed if and only if every sequence of points in A that converges in (X,d) converges to a point in A.
   (c) Prove that any closed ball is a closed set.
   (d) Let $y_1, y_2 \in X$ and $r \in \mathbb{R}$. Define

$$H(y_1, y_2, r) = \{x \in X : d(x, y_1) - d(x, y_2) < r\}.$$

   Prove that $H(y_1, y_2, r)$ is an open set.

5. (a) Define a *contraction mapping* in a metric space.
   (b) State and prove the Contraction Mapping Theorem.
   (c) Let T be a contraction mapping with $d(Tx, Ty) \leq \frac{d(x,y)}{2}$ for each $x, y \in X$. Suppose, $z \in X$ is a point such that any other point from X lies within distance 1 from z. For which smallest n can we guarantee that $T^n(z)$ is within distance 10^{-3} from the fixed point of T?
   (d) Prove that the system of equations

$$\sin(4y) + 5 - x = 0$$

$$\cos(x/5) - 3 - y = 0$$

   has a unique real solution (x, y).

6. (a) Define the *operator norm* $\|T\|$ of a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$. Prove that if $S : \mathbb{R}^n \to \mathbb{R}^n$ is linear then $\|S + T\| \leq \|S\| + \|T\|$ and $\|ST\| \leq \|S\|\|T\|$
   (b) Prove that if a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\|Ax\|_2 \geq 2\|x\|_2$ for all $x \in \mathbb{R}^n$ then $I + A$ is invertible. [Hint: you may wish to use the fact that if $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\|T\| < 1$, then $I - T$ is invertible, but you would have to prove this.]

END OF PAPER