# UNIVERSITY COLLEGE LONDON

University of London

## EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

*B.Sc.* *M.Sci.*

**Mathematics M212: Analysis 4: Real Analysis**

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| COURSE CODE | : | **MATHM212** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **24-MAY-05** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 Hours** |

05-C0962-3-110


**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Define what it means for a sequence of functions $\{f_n\}_{n=1}^{\infty}$ to converge *uniformly* on an interval $[a,b]$.

   (b) Prove that if $f_n \to f$ uniformly on $[a,b]$, and each $f_n$ is continuous on $[a,b]$, then $f$ is continuous on $[a,b]$.

   (c) Prove that if $f_n \to f$ pointwise on $[a,b]$, $f_n$ and $f$ are continuous functions and for each $x \in [a,b]$ the sequence $\{f_n(x)\}$ is monotone, then $f_n \to f$ uniformly.

   (d) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of differentiable functions on $\mathbb{R}$ such that $f_n(x) \to 0$ pointwise and $f_n'(x) \to 0$ pointwise. Must $f_n \to 0$ uniformly?

2. (a) State the Weierstrass $M$-test.

   (b) Prove that there exists a continuous function on the real line which is nowhere differentiable.

3. (a) Define the *Fourier series* $\sum_{n=1}^{\infty} a_n \phi_n$ of a Riemann integrable function $f$ on an interval $[a,b]$ with respect to an orthonormal system $(\phi_n)_{n=1}^{\infty}$.

   (b) Show that

$$\sum_{n=1}^{\infty} a_n^2 \leq \int_a^b f(x)^2 dx.$$

   (c) By considering the Fourier series of $f(x) = x$ with respect to an appropriate *orthonormal* system on $[-\pi, \pi]$, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{\pi^2}{6}.$$

4. (a) Define the terms metric space, open ball, closed ball, open set and closed set.
   (b) Let $(X, d)$ be a metric space. Prove that $A \subset X$ is closed if and only if every sequence of points in $A$ that converges in $(X, d)$ converges to a point in $A$.
   (c) Prove that if $(X, d)$ is a metric space and $(F_j)_{j \in J}$ is a collection of closed subsets of $X$ then $\cap_{j \in J} F_j$ is closed.
   (d) Prove that if $(X, d)$ is a metric space, $n$ is a positive integer, and $(F_j)_{j=1}^{n}$ are closed subsets of $X$ then $\cup_{j \in J} F_j$ is closed.
   (e) If $(X, d)$ is a metric space and $(F_j)_{j=1}^{\infty}$ are closed subsets of $X$ then must $\cup_{j=1}^{\infty} F_j$ be closed?

5. (a) Define a *contraction mapping* in a metric space.
   (b) State and prove the Contraction Mapping Theorem.
   (c) Give examples showing that the condition of the mapping $T$ being a contraction mapping in (b) cannot be replaced by the condition $d(Tx, Ty) < d(x, y)$.
   (d) Prove that the equation
   $$\sin(x/4) + 4 - x = 0$$
   has a unique real solution.

6. (a) Define the *operator norm* $\|T\|$ of a linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Prove that if $S : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ is linear then $\|ST\| \leq \|S\| \|T\|$.
   (b) Prove that if a linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has norm $\|T\| < 1$ then $I - T$ is invertible.
   (c) Let $S : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be given by $S(x_1, \ldots, x_4) = x_1 - 2x_2 + 3x_3 - 4x_4$. Find $\|S\|$.