UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M212: Analysis 4: Real Analysis

COURSE CODE	:	**MATHM212**
UNIT VALUE	:	**0.50**
DATE	:	**14-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

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TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define what it means for a sequence of functions $(f_n(x))_{n=1}^{\infty}$ to converge *uniformly* on an interval $[a, b]$.

   (b) Prove that if $f_n(x) \to f(x)$ uniformly on $[a, b]$, and each f_n is continuous on $[a, b]$, then f is continuous on $[a, b]$.

   (c) Prove that if $f_n(x) \to f(x)$ uniformly on $[a, b]$, and each f_n is Riemann integrable, then f is Riemann integrable and $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$ as $n \to \infty$.

   (d) Suppose that $(f_n(x))_{n=1}^{\infty}$ is a sequence of continuous functions on $[0, 1]$ such that $f_n(x) \to 0$ pointwise and $\int_0^1 f_n(x)dx = 0$ for every n. Must $f_n(x) \to 0$ uniformly?

2. (a) Define what it means for a function $f(x)$ to be *uniformly continuous* on $[0, 1]$.

   (b) Prove that every continuous function on $[0, 1]$ is uniformly continuous.

   (c) Prove that if $f(x)$ is continuous on $[0, 1]$ then there is a sequence of polynomials that converges uniformly to $f(x)$ on $[0, 1]$.

   [You may assume standard facts about the polynomials $p_{nk}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, provided that you state them clearly.]

3. (a) Define the *Fourier series* $\sum_{n=1}^{\infty} a_n\phi_n$ of a Riemann integrable function f on an interval $[a, b]$ with respect to an orthonormal system $(\phi_n)_{n=1}^{\infty}$.

   (b) Show that $\sum_{n=1}^{\infty} a_n^2 \leqslant \int_a^b f(x)^2dx$. Deduce that $a_n \to 0$ as $n \to \infty$.

   (c) By considering the Fourier series of $f(x) = x$ with respect to an appropriate *orthonormal* system on $[-\pi, \pi]$, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leqslant \frac{\pi^2}{6}.$$

4. (a) Define what it means for a function $d : X \times X \to \mathbb{R}$ to be a *metric*. Define the terms *open ball* and *open set*.

(b) Let (X, d_X) and (Y, d_Y) be metric spaces. Define what it means for a function $f : X \to Y$ to be *continuous*. Prove that if $f : X \to Y$ is continuous and G is an open subset of Y then $f^{-1}(G)$ is open in X.

(c) Define what it means for a set K in a metric space to be *compact*. Prove that a continuous image of a compact set is compact.

(d) Prove that if K is compact then every sequence of points of K has a convergent subsequence.

(e) Consider the space $X = (C[0,1], ||.||_1)$ of continuous functions on $[0,1]$ with the norm

$$||f||_1 = \int_0^1 |f(x)|dx.$$

Prove that the closed unit ball of X is not compact.

5. (a) Define a *contraction mapping* $f : X \to X$ in a metric space (X, d).

(b) State and prove the Contraction Mapping Theorem.

(c) Suppose that $0 \leqslant c < 1$, and $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are functions such that, for every x_1, x_2, y_1, y_2,

$$|f_1(x_1, y_1) - f_1(x_2, y_2)| \leqslant c|x_1 - x_2|$$

and

$$|f_2(x_1, y_1) - f_2(x_2, y_2)| \leqslant c|y_1 - y_2|.$$

Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = (f_1(x, y), f_2(x, y))$$

has a unique fixed point.

6. (a) Define the operator norm $||T||$ of a linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Prove that if $S : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ is linear then $||ST|| \leqslant ||S|| \cdot ||T||$.

(b) Prove that if T is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $||T|| < 1$ then $I - T$ is invertible. [You may assume that the space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ is complete with the operator norm.]

(c) Define what it means for a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ to be *differentiable* at $\mathbf{x} \in \mathbb{R}^n$. Define the derivative $Df(\mathbf{x})$ and the Jacobian matrix $Jf(\mathbf{x})$ at a point $\mathbf{x} \in \mathbb{R}^n$.

(d) Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $\mathbf{x}$. Prove that the partial derivatives $\frac{\partial f_i}{\partial x_j}(\mathbf{x})$ exist and that $Df(\mathbf{x})$ can be represented by $Jf(\mathbf{x})$.

(e) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = \frac{x^2 y}{x^4 + y^2}$$

for $(x, y) \neq (0, 0)$, and

$$f(0, 0) = 0.$$

Show that the Jacobian matrix exists at the point $(0, 0)$, but that f is not differentiable at $(0, 0)$.