UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M212: Analysis 4: Real Analysis

COURSE CODE	:	**MATHM212**
UNIT VALUE	:	**0.50**
DATE	:	**27-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0949-3-110

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define what it means for a sequence of functions $(f_n(x))_{n=1}^{\infty}$ to converge *uniformly* on an interval $[a, b]$.

   (b) Prove that if $f_n(x) \to f(x)$ uniformly on $[a, b]$, and each f_n is continuous on $[a, b]$, then f is continuous on $[a, b]$.

   (c) Prove that if $f_n(x) \to f(x)$ uniformly on $[a, b]$, and each f_n is Riemann integrable, then f is Riemann integrable and $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$ as $n \to \infty$.

   (d) Suppose that $f_n(x) \to f(x)$ pointwise on $[a, b]$, and that f and each f_n are Riemann integrable. Suppose also that $\int_a^b f_n(x)dx = 0$ for each n and $\int_a^b f(x)dx = 0$. Must $f_n(x) \to f(x)$ uniformly?

2. (a) State the Weierstrass M-test.

   (b) Prove that there exists a real continuous function on the real line that is nowhere differentiable.

3. (a) Define what it means for a function $d : X \times X \to \mathbb{R}$ to be a *metric.* Define the terms *open ball, closed ball, open set* and *closed set.*

   (b) Let (X, d_X) and (Y, d_Y) be metric spaces. Define what it means for a function $f : X \to Y$ to be *continuous.* Prove that if $f : X \to Y$ is continuous then $f^{-1}(G)$ is open in X whenever G is open in Y.

   (c) Define what it means for a set K in a metric space to be *compact.* Prove that a continuous image of a compact set is compact.

   (d) Prove that a finite union of compact sets is compact.

   (e) Must the union of an infinite sequence $K_1, K_2, \ldots$ of compact sets be compact?

4. (a) Define a *contraction mapping* in a metric space.

(b) State and prove the Contraction Mapping Theorem.

(c) Suppose that $f : [0, \infty) \to [0, \infty)$ is a continuous function such that

$$|f(x) - f(y)| < |x - y|$$

for all $x, y \in [0, \infty)$ with $x \neq y$. Must f have a fixed point?

5. (a) Prove that if f is Riemann integrable on $[a, b]$ then

$$\lim_{\lambda \to \infty} \int_a^b f(x) \cos(\lambda x) dx = 0.$$

(b) Define the *Dirichlet kernel* $D_n(t)$.

(c) Suppose that f is Riemann integrable on $[-\pi, \pi]$. Define the Fourier coefficients of f and, for $n \geqslant 1$, the partial sum s_n of the Fourier series. Prove that, for $n \geqslant 1$,

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x - t) dt.$$

6. (a) Define the norm $||T||$ of a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$.

(b) Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be *differentiable* at $\mathbf{x} \in \mathbb{R}^n$.

(c) Define the Jacobian matrix $Jf(\mathbf{x})$ for a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ and prove that $Df(\mathbf{x})$ can be represented by $Jf(\mathbf{x})$.

(d) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has all partial derivatives $\frac{\partial f_i}{\partial x_j}$ at $\mathbf{x} \in \mathbb{R}^n$. Must f be differentiable at $\mathbf{x}$?

(e) State the Chain Rule for differentiable functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$. If f is invertible and f^{-1} is differentiable at $\mathbf{y} = f(\mathbf{x})$, what is $Df^{-1}(\mathbf{y})$?