UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M212: Analysis 4: Real Analysis

COURSE CODE	:	**MATHM212**
UNIT VALUE	:	**0.50**
DATE	:	**16-MAY-02**
TIME	:	**10.00**
TIME ALLOWED	:	**2 hours**

02-C0944-3-90

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define what it means for a sequence of functions $(f_n)_{n=1}^{\infty}$ to converge *uniformly* on an interval $[a, b]$.

   (b) *State* the Heine-Borel Theorem.

   (c) Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous real functions on $[a, b]$ that converges pointwise to a continuous function f. Suppose in addition that $(f_n(x))_{n=1}^{\infty}$ is monotonic for each $x \in [a, b]$. Prove that $f_n \to f$ uniformly on $[a, b]$.

   (d) Suppose that $f_n \to 0$ pointwise on $[a, b]$ and $(|f_n(x)|)_{n=1}^{\infty}$ is decreasing for every x. Must $f_n \to 0$ uniformly?

2. (a) Define what it means to say that a sequence of functions $(f_n)_{n=1}^{\infty}$ on a set $I \subseteq \mathbb{R}$ is a *uniform Cauchy sequence.*

   (b) State and prove the General Principle of Uniform Convergence for a sequence of functions $(f_n)_{n=1}^{\infty}$ on a set $I \subseteq \mathbb{R}$.

   (c) Suppose that $\sum_{n=1}^{\infty} M_n$ is a convergent series of nonnegative real numbers and $(f_n)_{n=1}^{\infty}$ is a sequence of functions on $I \subset \mathbb{R}$ such that $||f_n||_{\text{sup}} \leq M_n$ for every n. Prove that the *series* $\sum_{n=1}^{\infty} f_n$ converges uniformly.

   (d) Suppose that $\sum_{n=1}^{\infty} f_n$ is a uniformly convergent series of functions on $I \subseteq \mathbb{R}$. Prove that $||f_n||_{\text{sup}} \to 0$ as $n \to \infty$.

   (e) Suppose $\sum_{n=1}^{\infty} f_n$ is a uniformly convergent series of functions. Must $\sum_{n=1}^{\infty} ||f_n||_{\text{sup}}$ converge?

3. Let $(\phi_n)_{n=1}^{\infty}$ be an orthonormal sequence of functions on $[a, b]$ and suppose that f is Riemann integrable on $[a, b]$.

   (a) Define the *Fourier series* $\sum_{n=1}^{\infty} a_n \phi_n$ of f on an interval $[a, b]$ with respect to $(\phi_n)_{n=1}^{\infty}$.

   (b) Show that, for $n \geq 1$,
   $$\sum_{i=1}^{n} a_i^2 \leq \int_a^b f(x)^2 dx.$$

   (c) Prove that for $n \geq 1$ and any real numbers $c_1, \ldots, c_n$,
   $$\int_a^b (f(x) - \sum_{i=1}^{n} a_i \phi_i(x))^2 dx \leq \int_a^b (f(x) - \sum_{i=1}^{n} c_i \phi_i(x))^2 dx.$$

(d) Determine the Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

for the function $f(x) = |x|$ on $[-\pi, \pi]$. Assuming that this converges to $f(x)$ at 0, deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

4. (a) Define a *contraction mapping* in a metric space.

(b) State and prove the Contraction Mapping Theorem.

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are contraction mappings. Prove that there are points $x, y \in \mathbb{R}$ such that $f(x) = y$ and $g(y) = x$. [Hint: Consider the mapping $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x, y) = (g(y), f(x))$. Choose a suitable metric.]

5. (a) Define what it means for a function $d : X \times X \to \mathbb{R}$ to be a *metric*. Define the terms *open ball* and *open set* in a metric space.

(b) Define what it means for a function f between two metric spaces (X, d_X) and (Y, d_Y) to be *continuous*. Prove that if $f : X \to Y$ is continuous then $f^{-1}(G)$ is open in X whenever G is open in Y.

(c) Define what it means for a set K in a metric space to be *compact*. Prove that a continuous image of a compact set is compact.

(d) Prove that if K is compact then every sequence of points of K has a convergent subsequence.

(e) Consider the space $X = (C[0,1], ||.||_{\text{sup}})$ of continuous functions on $[0,1]$ with the supremum norm. Prove that the closed unit ball of X is not compact.

6. (a) Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be *differentiable* at $\mathbf{x} \in \mathbb{R}^n$.

(b) Prove that if $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear then $DT(\mathbf{x}) = T$ for all $\mathbf{x} \in \mathbb{R}^n$.

(c) Define the Jacobian matrix $Jf(\mathbf{x})$ for a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$. Prove that $Df(\mathbf{x})$ can be represented by $Jf(\mathbf{x})$.

(d) State the Chain Rule for differentiable functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$.

(e) Suppose $f, g : \mathbb{R}^2 \to \mathbb{R}$ are differentiable functions. Let $u(r, \theta) = f(r\cos\theta, r\sin\theta)$ and $v(r, \theta) = g(r\cos\theta, r\sin\theta)$. Use the Chain Rule to write down expressions for $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, where $x = r\cos\theta$ and $y = r\sin\theta$. [Hint: Let $h(r, \theta) = (r\cos\theta, r\sin\theta)$ and consider the composition $f \circ h$.]

END OF PAPER