## **UNIVERSITY COLLEGE LONDON**

1

## University of London

## **EXAMINATION FOR INTERNAL STUDENTS**

For The Following Qualifications:-

B.Sc.

M.Sci.

Mathematics M211: Analysis 3: Complex Analysis

COURSE CODE

: MATHM211

UNIT VALUE

: 0.50

DATE

: 05-MAY-04

TIME

: 14.30

TIME ALLOWED

: 2 Hours

All questions may be attempted but only marks obtained on the best four solutions will count. The use of an electronic calculator is not permitted in this examination.

- 1. (i) Let f be a complex valued function of z = x + iy, x, y real. Write f(z) = u(x, y) + iv(x, y), u(x, y), v(x, y) real. If f is differentiable at z, show that the Cauchy-Riemann equations  $u_x = v_y$ ,  $v_x = -u_y$  hold at z.
  - (ii) Let  $f(z) = |z|^2$ . Show that f is differentiable only at 0.
  - (iii) Let

$$g(z) = \frac{z^3}{|z|^2}, \quad z \neq 0.$$
  
 $g(0) = 0.$ 

Show that g is not differentiable at 0.

- 2. (i) Let f be differentiable on the open disc D(0,R) of centre 0 and radius R. Show that
  - a) if  $f'(z) = 0 \ \forall z \in D(0, R)$ , then f is constant,
  - b) if |f(z)| is constant on D(0,R) then f is constant.
  - (ii) If  $\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$  show that

$$\exp(a+b) = \exp a. \exp b, \quad \forall a, b \in \mathbb{C}.$$

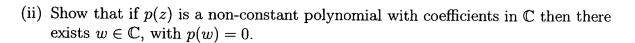
- (iii) If  $\sin z = 1$ , show that z is a real number.
- 3. (i) State and prove Taylor's theorem for a function f differentiable in the open disc D(a, R), centre a, radius R.
  - (ii) Find the Taylor expansion of  $(1-z)^{-1}$  of the form  $\sum_{n=0}^{\infty} c_n(z+i)^n$ , valid in the disc  $D(-i,\sqrt{2})$ .

Would the expansion be valid in the disc D(-i, 2)?

(iii) Find the Laurent expansion, about 0, of  $(z^3-1)^{-1}$  valid a) for |z|<1, b) |z|>1.

MATHM211

4. (i) If f is holomorphic and bounded on  $\mathbb{C}$ , show that f is constant on  $\mathbb{C}$ .



(iii) If 
$$q(z) = 7z^7 + 8z^6 + 3z^5 + 4z^4 + 1$$
, show that 
$$\sup\{|q(z)|; |z| = 2\} > \sup\{|q(z)|; |z| = 1\}.$$

5. (i) Let f be a continuous function defined in the upper half plane Im  $z \ge 0$ , where  $f(z) \to 0$  uniformly as  $|z| \to \infty$ . If m > 0, show that

$$\int_{\Gamma} e^{imz} f(z) dz \to 0 \text{ as } R \to \infty$$

where  $\Gamma_R = \left\{ \operatorname{Re}^{i\theta}, \ 0 \leqslant \theta \leqslant \pi \right\}$ .

(ii) Evaluate

$$\int_0^\infty \frac{\sin x}{x(x^2+1)} dx.$$

6. (i) Let f be holomorphic inside and on a convex contour  $\gamma$  except for a finite number of poles at  $a_1, ..., a_n$  inside  $\gamma$  with residues  $R_1, ..., R_n$  respectively. Show that

$$\int_{\gamma} f(z)dz = 2\pi i (R_1 + \dots + R_n).$$

(ii) Prove that

$$\int_0^{2\pi} \cos^{2n} x + \sin^{2n} x dx = \left(\frac{1}{2}\right)^{2n-2} \binom{2n}{n} \pi;$$

n a positive integer.