UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M120: Algebra for Joint Honours Students

COURSE CODE	:	**MATHM120**
UNIT VALUE	:	**0.50**
DATE	:	**23-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1036-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 4 & 6 & 1 & 2 & 5 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 2 & 6 & 7 & 5 \end{pmatrix}.$$

   (a) Express σ and τ in cycle notation.
   (b) Find (in standard notation) $\tau\sigma$, $\sigma\tau$, and τ^{-1}.
   (c) What are the orders of τ, σ, and $\sigma\tau$?
   (d) Give, with explanations, the signs of τ, σ, and $\sigma\tau$.
   (e) Express the 7-cycle (1234567) as a product of 2-cycles.
   (f) Let S_n be the group of permutations of n elements. What is the order of S_7? Find an isomorphism between S_7 and a subgroup of S_8.

2. (a) Let $\langle \mathbb{Z}_7^*, \bullet \rangle$ be the multiplicative group mod 7 (where $a \bullet b \equiv ab \mod 7$). In this group, find the group elements corresponding to
      (i) $5 \bullet 6$;
      (ii) 3^{-1};
      (iii) 4^{-1};
      (iv) 6^{11}.
   (b) Find the following numbers using modular arithmetic:
      (i) $-33 \mod 7$;
      (ii) $18^{399964} \mod 17$;
      (iii) $724599364 \mod 9$;
      (iv) $22^{301} \mod 31$.
   (c) For each of the following functions, determine which functions are one-to-one (injective), and which functions are onto (surjective). If a function is not one-to-one or not onto, explain why not.
      (i) $f : \mathbb{R} \to \mathbb{R}$, $\quad f(x) = \tanh x$;
      (ii) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $\quad g(x) = (x-1)^3$;
      (iii) $h : \mathbb{Z} \to \mathbb{Z}$, $\quad h(x) = 2x - 1$.
   (d) Consider the symmetry group of a square, D_4, with elements $\{e, r, r^2, r^3, f, rf, r^2f, r^3f\}$.
      (i) Show by simple diagrams, or otherwise, that
$$fr = r^3 f.$$
      (ii) Reduce the following element to its simplest form:
$$f^{17} r^{33} f^{92} r^{28} f^{11}.$$

3. (a) What are the four defining properties of a group?

   (b) Suppose $\langle G, *\rangle$ is a group, and let centre(G) be the set of elements in G which commute with all other elements in G: i.e. if $c \in$ centre(G) then

$$cg = gc \qquad \forall g \in G.$$

   Show that centre(G) is a subgroup of G.

   (c) Let $\mathcal{C}$ be the set of complex numbers z of unit modulus $|z| = 1$, i.e. complex numbers of the form $z = \exp i\theta$.

      (i) Show that $\mathcal{C}$ is a group under complex multiplication.

      (ii) Show that the set of fifth roots of unity

$$\mathcal{W}_5 = \{w \in \mathbb{C} \mid w^5 = 1\}$$

      with complex multiplication is a subgroup of $\mathcal{C}$.

      (iii) Find an isomorphism between the subgroup $\mathcal{W}_5$ and $\langle \mathbb{Z}_5, \oplus\rangle$, the additive group mod 5 (where $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and $a \oplus b \equiv a + b \mod 5$).

4. (a) Consider row reduction operations on a 3×3 matrix. What is the elementary matrix corresponding to exchanging rows 2 and 3? What is the elementary matrix which adds row 1 to row 2?

   (b) Briefly describe how to invert matrices using elementary row operations.

   (c) What is the augmented matrix corresponding to the system of linear equations

$$\begin{aligned} 2x + 3y - z &= -1; \\ x + y + z &= 6; \\ -2x + y + 2z &= 1. \end{aligned}$$

   (d) Transform the augmented matrix to reduced row echelon form. What is the solution (if any) to the system of equations?

5. (a) State what it means for $T : U \to V$ to be a linear transformation between vector spaces U and V. Define the kernel $Ker(T)$ and image $Image(T)$ of such a transformation.

(b) State (without proof) a formula involving the dimensions of these spaces.

(c) Let U be the space of real polynomials of degree at most 4. What is the dimension of U? Define $T : U \to U$ by the following: for a function $p(x)$ in U,

$$(Tp)(x) = \frac{\mathrm{d}^2 p(x)}{\mathrm{d}x^2}.$$

Find the kernel and image of T. What are their dimensions?

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ into $T\mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and maps $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ into $T\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Use the fact that T is linear to find $T(2\mathbf{a}+\mathbf{b})$. What matrix (with respect to the standard basis) does T correspond to?

(e) Are the following three vectors linearly independent?

$$\begin{aligned} \mathbf{v}_1 &= (1, 0, -1); \\ \mathbf{v}_2 &= (0, 1, -1); \\ \mathbf{v}_3 &= (1, -1, 0). \end{aligned}$$

6. Let

$$A = \begin{bmatrix} 4 & 1 \\ -5 & -2 \end{bmatrix}$$

(a) Find all eigenvalues for A.

(b) Find associated eigenvectors.

(c) Find an invertible matrix B and a diagonal matrix Δ such that $B^{-1}AB = \Delta$.

(d) Find an expression for A^{17} in terms of B and Δ (you do not need to calculate this expression).