UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M221: Algebra 3: Further Linear Algebra

COURSE CODE : **MATHM221**

UNIT VALUE : **0.50**

DATE : **17-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-C1045-3-180

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Find integers h and k such that

$$9h + 11k = 1.$$

(b) State and prove the Chinese Remainder Theorem.

(c) Find the unique $x \in \mathbb{Z}/99$ such that

$$x \equiv 3 \bmod 9 \quad \text{and} \quad x \equiv 7 \bmod 11.$$

(d) Prove that there are infinitely many prime numbers congruent to 2 modulo 3.

2. (a) (i) Find $\mathrm{hcf}(f, g)$ in $\mathbb{Q}[X]$ when $f(X) = X^3 - 1$ and $g(X) = X^2 + 1$.

(ii) Find $h, k \in \mathbb{Q}[X]$ such that $\mathrm{hcf}(f, g) = hf + kg$.

(b) Let k be a field and $f \in k[X]$. Define the following:

(i) f is irreducible,

(ii) f is monic,

(iii) $\deg(f)$.

(c) Let $p \in k[X]$ be an irreducible polynomial. Prove that if $p|fg$ then $p|f$ or $p|g$.

(d) Suppose that $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ are irreducible monic polynomials such that

$$p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s.$$

Prove that $r = s$ and (after reordering if necessary) $p_i = q_i$ for $i = 1, \ldots, r$.

3. (a) Define the *minimal polynomial* of a linear map.

   (b) Prove that the minimal polynomial exists and is unique.

   (c) Explain what is meant by a *generalized eigenspace* and state (but do <u>not</u> prove) the *primary decomposition theorem.*

   (d) Prove that a matrix is diagonalizable if and only if its minimal polynomial is a product of distinct linear factors.

   (e) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

   Hence determine whether A is diagonalizable over $\mathbb{Q}$.

4. (a) Define the following terms.

      (i) Jordan block matrix.

      (ii) Jordan basis.

   (b) In each of the following cases find the Jordan normal form of the matrix A.

      (i) $\mathrm{ch}_A(X) = (X-1)^2$, $m_A(X) = X - 1$.

      (ii) $\mathrm{ch}_A(X) = m_A(X) = (X-3)^{10}$.

      (iii) $\mathrm{ch}_A(X) = (X-6)^3$, $m_A(X) = (X-6)^2$.

      (iv) $\mathrm{ch}_A(X) = (X-1)^{10}(X-2)^{10}$, $m_A(X) = (X-1)^6(X-2)^4$, $\mathrm{null}(A - I) = 4$, $\mathrm{rank}(A - 2I) = 16$ and $\mathrm{null}((A-2I)^2) = 8$.

   (c) Let V be the vector space of polynomials of degree at most two

$$V = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{C}\}.$$

   Let $\alpha : V \longrightarrow V$ be the linear map defined by

$$\alpha(f) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - 4f.$$

   Find the Jordan canonical form of α.

5. (a) State and prove Sylvester's Law of Inertia.

   (b) Consider the following quadratic forms:

$$q_1\begin{pmatrix}x\\y\\z\end{pmatrix} = x^2 + 2xy + 2xz + 3y^2 + 2yz + 2z^2,$$

$$q_2\begin{pmatrix}x\\y\\z\end{pmatrix} = 2x^2 + 8xy + 4xz + 2y^2 + 8yz + 6z^2.$$

   By finding their canonical forms, determine whether q_1 and q_2 are

   (i) congruent over $\mathbb{R}$;

   (ii) congruent over $\mathbb{C}$.

6. (a) Let V be a vector space over $\mathbb{C}$. What does it mean to say that $\langle \cdot, \cdot \rangle$ is a positive definite Hermitian form on V?

   (b) Using the Gram–Schmidt process, find an orthonormal basis for the following vector space with respect to the given inner product:

$$V = \{a + bx : a, b \in \mathbb{C}\}, \quad \langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

   (c) Let V be a vector space over $\mathbb{C}$ with a positive definite Hermitian inner product and let $T : V \to V$ be a self-adjoint linear map.

   (i) Show that if $U \subseteq V$ is an invariant subspace, then its orthogonal complement, $U^\perp$, is also invariant.

   (ii) State and prove the Spectral Theorem.