UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M221: Algebra 3: Further Linear Algebra

COURSE CODE	:	**MATHM221**
UNIT VALUE	:	**0.50**
DATE	:	**18-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0963-3-170

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) (i) Prove that there exist infinitely many prime numbers.
   (ii) Show that if p_k is the k^{th} prime then $p_k < 2^{2^k}$. (You may wish to use induction on k.)
   (b) State Fermat's Little Theorem and calculate
   (i) $4^{44} \mod 47$,
   (ii) $4^{29}5^{60} \mod 61$.
   (c) If $a \geqslant 2$ and $p \geqslant 2$ are integers satisfying $a^{p-1} \equiv 1 \mod p$ calculate $\gcd(a, p)$.

2. (a) Let $\mathbb{F}$ be a field. Prove that if $a, b \in \mathbb{F}[x]$ and $b \neq 0$ then there exist unique polynomials $q, r \in \mathbb{F}[x]$ such that $\deg(r) < \deg(b)$ and $a = bq + r$.
   (b) (i) If $f, g \in \mathbb{F}[x]$ define $\gcd(f, g)$.
   (ii) Find $\gcd(f, g)$ when $f(x) = x^4 - 1$ and $g(x) = x^2 + x - 2$ belong to $\mathbb{Q}[x]$.
   (iii) Find $h, k \in \mathbb{Q}[x]$ such that $\gcd(f, g) = ha + kb$.
   (c) Find the minimal polynomial of the matrix

$$A = \begin{bmatrix} 2 & 6 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   Calculate

$$B = A^{102} - 4A^{101} + 4A^{100} + A - I.$$

3. (a) Let V be a vector space over a field $\mathbb{F}$. Define what it means to say that $f : V \times V \longrightarrow \mathbb{F}$ is a symmetric bilinear form.

   (b) Let A be a real symmetric matrix. Prove that if A is congruent to both $B = \operatorname{diag}(I_p, -I_q, 0)$ and $C = \operatorname{diag}(I_j, -I_k, 0)$ then $B = C$.

   (c) Find the real and complex canonical forms of the following matrices

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

   Hence or otherwise decide whether A and B are

   (i) congruent over $\mathbb{R}$,

   (ii) congruent over $\mathbb{C}$.

4. (a) Let V be a vector space over $\mathbb{C}$. What does it mean to say that $<,>$ is an inner product on V?

   (b) (i) Define what it means to say that $A \in M_n(\mathbb{C})$ is hermitian.

   (ii) Prove that if $\lambda \in \mathbb{C}$ is an eigenvalue of a hermitian matrix $A \in M_n(\mathbb{C})$ then λ is real.

   (iii) State the Spectral Theorem.

   (iv) Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

   Find an orthogonal matrix P such that $P^T AP = P^{-1}AP$ is diagonal.

5. (a) (i) State the Primary Decomposition theorem (giving definitions of the minimal polynomial and generalised eigenspaces).

(ii) Prove that if V is a finite dimensional vector space over a field $\mathbb{F}$ and $\alpha : V \to V$ is a linear map then α is diagonalisable iff $m_\alpha(x)$ is a product of distinct linear factors.

(b) Let V be the real vector space of functions with basis

$$\mathcal{E} = \{\cos x, \sin x, \cos 2x, \sin 2x\}.$$

Define $D : V \to V$ by $D(f) = \mathrm{d}f/\mathrm{d}x$.

(i) Calculate $[D]_{\mathcal{E}}$, the matrix representing D with respect to $\mathcal{E}$.

(ii) Express the minimal polynomial of D as the product of monic irreducibles and decide whether D is diagonalisable.

(iii) Let W be the complex vector space with the same basis $\mathcal{E}$. Is $D : W \to W$ diagonalisable?

6. (a) In each of the following cases find the Jordan normal form of the matrix A.

(i) $c_A(x) = x^3$, $m_A(x) = x$.

(ii) $c_A(x) = m_A(x) = (x-2)^2$.

(iii) $c_A(x) = (x-4)^3$, $m_A(x) = (x-4)^2$.

(iv) $c_A(x) = (x-1)^{10}(x-2)^{10}$, $m_A(x) = (x-1)^5(x-2)^5$, $\mathrm{null}(A-I) = 5$, $\mathrm{rank}(A-2I) = 16$ and $\mathrm{null}((A-2I)^2) = 7$.

(v)

$$A = \mathrm{diag}\left(\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

(b) (i) Show that if A and B are 3×3 complex matrices with the same characteristic and minimal polynomials then they have the same Jordan normal form.

(ii) Is the same true of 4×4 complex matrices? (Give a proof or counterexample.)