UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M221: Algebra 3: Further Linear Algebra

COURSE CODE	:	**MATHM221**
UNIT VALUE	:	**0.50**
DATE	:	**12-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0957-3-160

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let V be a vector space over a field $\mathbb{F}$. Define what it means to say that $f : V \times V \longrightarrow \mathbb{F}$ is a symmetric bilinear form.
Show that if $q : V \longrightarrow \mathbb{F}$ is a quadratic form and $1 + 1 \neq 0$ in $\mathbb{F}$ then there is a unique symmetric bilinear form f such that, for every $\mathbf{v} \in V$,

$$f(\mathbf{v}, \mathbf{v}) = q(\mathbf{v}).$$

Give an example of a quadratic form $q : \mathbb{F}_2 \longrightarrow \mathbb{F}_2$ for which this is not true.

(b) Consider the following symmetric matrices:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 2 & 2 \\ 4 & 2 & 5 \end{bmatrix}.$$

By finding their canonical forms decide whether A and B are

(i) congruent over $\mathbb{R}$;

(ii) congruent over $\mathbb{C}$.

2. (a) If V is a Euclidean space and $\alpha : V \longrightarrow V$ is a linear mapping define the adjoint of α.
What does it mean to say that α is self-adjoint?
Prove that if α is self-adjoint and $\mathcal{E}$ is an orthonormal basis for V then the matrix representing α with respect to $\mathcal{E}$ is symmetric.

(b) Show that if $A \in {}^n\mathbb{R}^n$ is symmetric then the eigenvalues of A are real.
Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

with characteristic polynomial $c_A(x) = x^2(x-3)$. Find an orthogonal matrix P such that $P^{-1}AP = P^TAP$ is diagonal.

3. (a) If V is a real vector space what does it mean to say that

   (i) $<,>$ is an inner product on V;

   (ii) $S \subset V$ is orthonormal (with respect to an inner product $<,>$).

   Show that if V is an inner product space and

$$S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\} \subset V$$

   is orthonormal then S is linearly independent.

   (b) Consider the inner product space of real polynomials of degree at most two

$$\mathcal{P}_2(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\},$$

   with the inner product

$$<f, g>= \int_{-1}^{1} f(x)g(x)\mathrm{d}x.$$

   Given that $\mathcal{E} = \{1, x, x^2\}$ is a basis for $\mathcal{P}_2(\mathbb{R})$, find an orthonormal basis $\mathcal{F}$ for $\mathcal{P}_2(\mathbb{R})$ by applying the Gram-Schmidt process.

   Find the transition matrix from $\mathcal{E}$ to $\mathcal{F}$.

4. (a) Consider the following polynomials $f, g \in \mathbb{Q}[x]$:

$$f(x) = x^4 + x^3 + x + 1, \qquad g(x) = x^2 - 1.$$

   (i) Using Euclid's algorithm find $d = \mathrm{hcf}(f, g)$.

   (ii) Find $h, k \in \mathbb{Q}[x]$ such that $hf + kg = d$.

   (b) Let V be a finite dimensional vector space over a field $\mathbb{F}$ and $\alpha : V \to V$ be a linear map with distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ and minimal polynomial

$$m_\alpha(x) = \prod_{i=1}^{r} (x - \lambda_i)^{b_i}.$$

   Define the term generalized eigenspace.

   State the Primary Decomposition theorem.

   Prove that if $f, g \in \mathbb{F}[x]$ and $\mathrm{hcf}(f, g) = 1$ then

$$\ker(fg(\alpha)) = \ker(f(\alpha)) \oplus \ker(g(\alpha)).$$

   Hence prove the Primary Decomposition theorem.

5. (a) Let V be an n-dimensional vector space over $\mathbb{C}$ and let $\alpha : V \longrightarrow V$ be a linear map with characteristic polynomial $c_\alpha(x) = (x - \lambda)^n$ and minimal polynomial $m_\alpha(x) = (x - \lambda)^b$. Define the following terms:

   (i) a Jordan basis;

   (ii) a Jordan block matrix, $J_n(\lambda)$.

   Describe the Jordan normal form of α, explaining how the number of blocks of each size may be calculated given the dimensions of the generalized eigenspaces.

   (b) In each of the following cases find the Jordan normal form of the matrix A.

   (i) $c_A(x) = (x-3)^3$, $m_A(x) = (x-3)$.

   (ii) $c_A(x) = (x-3)^3$, $m_A(x) = (x-3)^3$.

   (iii) $c_A(x) = (x-3)^4(x-2)^3$, $m_A(x) = (x-3)^2(x-2)^2$, $\text{null}(A-3I) = 2$.

   (iv) $c_A(x) = (x-5)^{32}(x-3)^{10}$, $m_A(x) = (x-5)^5(x-3)^8$, $\text{null}(A-5I) = 8$, $\text{null}((A-5I)^2) = 16$, $\text{null}((A-5I)^4) = 27$ and $\text{rank}(A-3I) = 40$.

   (v) $c_A(x) = (x-2)^{12}$, $(A-2I)^5 \neq 0$, $\text{rank}(A-2I) = 8$, $\text{null}((A-2I)^2) = 7$ and $\text{null}((A-2I)^4) = 10$.

6. (a) Suppose V is an n-dimensional vector space over $\mathbb{C}$ and $\alpha : V \longrightarrow V$ is a linear map with characteristic and minimal polynomials given by

$$c_\alpha(x) = m_\alpha(x) = (x-\lambda)^n.$$

   (i) Prove that if $\mathbf{v} \in V$ satisfies $(\alpha - \lambda 1)^{n-1}\mathbf{v} \neq \mathbf{0}$ then

$$\mathcal{E} = \{(\alpha - \lambda 1)^{n-1}\mathbf{v}, (\alpha - \lambda 1)^{n-2}\mathbf{v}, \ldots, (\alpha - \lambda 1)\mathbf{v}, \mathbf{v}\}$$

   is a basis for V.

   (ii) Calculate the matrix representing α with respect to $\mathcal{E}$.

   (b) Consider the following matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

   Find

   (i) J, the Jordan normal form of A;

   (ii) $P \in GL(3, \mathbb{C})$, satisfying $P^{-1}AP = J$;

   (iii) J^{20}.

   Find a matrix B such that $A^{20} = BP^{-1}$.

END OF PAPER