UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M221: Algebra 3: Further Linear Algebra

COURSE CODE	:	**MATHM221**
UNIT VALUE	:	**0.50**
DATE	:	**01-MAY-03**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

03-C0950-3-130

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let V be a finite dimensional vector space over $\mathbb{R}$ and let q be a quadratic form on V.
   (i) Explain what is meant by a canonical form of q.
   (ii) Prove the existence and uniqueness of the canonical form of q. (You may assume the existence of an orthogonal basis with respect to q).

   (b) Consider the following quadratic forms:

$$q_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + 2xz + 4yz, \quad q_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + 4xy + 2yz.$$

   Find the real and complex canonical forms of q_1 and q_2. Hence determine whether q_1 and q_2 are equivalent (i) over $\mathbb{R}$, (ii) over $\mathbb{C}$.

2. (a) Let V be a finite dimensional vector space with a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$.
   Define the *dual space* V^*.
   Define the *dual basis* $\mathcal{B}^* = \{b_1^*, \ldots, b_n^*\}$.
   Define the canonical map $V \to V^{**}$ and prove that this map is an isomorphism of vector spaces.

   (b) For a linear map $T : V \to W$, define the *adjoint* T^* of T.
   Show that $(T + U)^* = T^* + U^*$, where T and U are linear maps from V to W.
   Show that for $v \in V$ we have $T^{**}(v^{**}) = T(v)^{**}$, where T^{**} denotes the adjoint of T^*.

3. (a) Let V be a finite dimensional Euclidean space and let $T : V \to V$ be a self-adjoint linear map.

Show that for any eigenvector v of T,

$$T(\{v\}^{\perp}) \subseteq \{v\}^{\perp}.$$

Show that there is an orthonormal basis of V consisting of eigenvectors of T. (You may assume that all eigenvalues of T are real).

(b) Consider the following matrix:

$$A = \begin{pmatrix} -2 & -1 & 2 \\ -1 & -2 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

Find an orthogonal matrix M such that $M^{-1}AM$ is diagonal.

4. (a) Let V be an n-dimensional vector space and let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of V.

Show that for every n-linear alternating form f on V we have

$$f(v_1, \ldots, v_n) = f(b_1, \ldots, b_n) \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} b^*_{\sigma(i)}(v_i).$$

(You may assume that $f(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \operatorname{sign}(\sigma) f(v_1, \ldots, v_n)$.)

(b) Let V be an n-dimensional vector space and let f be a non-zero alternating n-linear form on V.

Show that $f(v_1, \ldots, v_n) = 0$ if and only if $\{v_1, \ldots, v_n\}$ is linearly dependent.

(c) Let $T : V \to V$ be a linear map.

Define the the determinant of T in terms of alternating forms on V.

Prove from the definition that $\det(T \circ U) = \det(T)\det(U)$.

5. (a) Consider the polynomials

$$f(x) = x^4 + 1, \qquad g(x) = x^3 + 1.$$

Find polynomials $p, q \in \mathbb{Q}[x]$ such that $pf + qg = 1$.

Suppose $\alpha \in \mathbb{C}$ satisfies $f(\alpha) = 0$. Find rational numbers a, b, c, d such that

$$\frac{1}{\alpha^3 + 1} = a\alpha^3 + b\alpha^2 + c\alpha + d.$$

(b) Let V be a vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear map.

Define the *minimal polynomial* m_T of T and prove that m_T is a factor of the characteristic polynomial ch_T. (You may assume the Cayley–Hamilton Theorem).

Prove that if λ is an eigenvector of T then $m_T(\lambda) = 0$.

6. (a) Explain carefully what it means for a matrix to be in Jordan canonical form.

(b) Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find (i) the characteristic polynomial of A; (ii) the minimal polynomial of A; (iii) the Jordan canonical form of A; (iv) a Jordan basis.

(c) Let $T : \mathbb{C}^7 \to \mathbb{C}^7$ be a linear map with characteristic polynomial $ch_T(X) = (X-1)^4(X-3)^3$ and minimal polynomial $m_T(X) = (X-1)^2(X-3)$.

(i) List all possibilities for the Jordan canonical form of T.

(ii) Given that $T - \mathrm{id}$ has nullity 3, identify the correct Jordan canonical form of T.