UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M221: Algebra 3: Further Linear Algebra

COURSE CODE	:	**MATHM221**
UNIT VALUE	:	**0.50**
DATE	:	**15-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0945-3-140

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let q be a quadratic form on a finite dimensional vector space V over $\mathbb{R}$. Define the following:

   (i) the symmetric bilinear form f corresponding to q;

   (ii) the matrix of f with respect to a basis $\mathcal{B}$ of V;

   (iii) the real canonical form of q.

   Prove that q cannot have two different real canonical forms.

   (b) Consider the following quadratic forms.

$$q_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + 2xy + 4xz + 2y^2 + 6yz, \quad q_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2x^2 + 4xy - z^2.$$

   Find the real and complex canonical forms of q_1 and q_2.

   Hence determine whether q_1 and q_2 are equivalent (i) over $\mathbb{R}$ and (ii) over $\mathbb{C}$.

2. (a) Let V be a finite dimensional vector space over a field k and let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of V.

   Define (i) the *dual space* V^* and (ii) the *dual basis* $\mathcal{B}^*$.

   Prove that for any $\varphi \in V^*$,

$$\varphi = \varphi(b_1)b_1^* + \ldots + \varphi(b_n)b_n^*.$$

   (b) Let $\mathcal{E} = \{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ and let $\mathcal{B} = \{b_1, b_2\}$ be the basis given by

$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

   Calculate $b_1^*(e_1)$, $b_2^*(e_1)$, $b_1^*(e_2)$ and $b_2^*(e_2)$.

   Express b_1^* and b_2^* as linear combinations of e_1^* and e_2^*.

   (c) Write down the canonical map $V \to V^{**}$ and prove that this map is an isomorphism of vector spaces.

3. (a) Define the terms *Euclidean space* and *orthonormal basis.*

(b) Let V be a Euclidean space and let $T : V \to V$ be a self-adjoint linear map.

Show that if $v \in V$ is an eigenvector of T, then $T(\{v\}^\perp) \subseteq \{v\}^\perp$.

Show that there is an orthonormal basis of V consisting of eigenvectors of T (you may assume that the eigenvectors of T are all real).

(c) Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 3 & 4 \\ 2 & 4 & 3 \end{pmatrix}.$$

Given that A has characteristic polynomial $(x+1)^2(x-8)$, find a basis $\mathcal{B}$ for $\mathbb{R}^3$, which is orthonormal with respect to the dot product, and whose elements are eigenvectors of A.

4. (a) Explain what is meant by an *n-linear alternating form.*

(b) Let f be an n-linear alternating form on an n-dimensional vector space V and let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for V.

Show that for any vectors $v_1, \ldots, v_n \in V$,

$$f(v_1, \ldots, v_n) = f(b_1, \ldots, b_n) D_{\mathcal{B}}(v_1, \ldots, v_n),$$

where $D_{\mathcal{B}}$ is given by

$$D_{\mathcal{B}}(v_1, \ldots, v_n) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} b^*_{\sigma(i)}(v_i).$$

(c) Let $T : V \to V$ be a linear map.

Define the *determinant* of T in terms of alternating forms on V.

Prove from the definition, that if the matrix of T with respect to $\mathcal{B}$ is $(a_{i,j})$, then

$$\det(T) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}.$$

(d) Given a permutation $\tau \in S_n$, define a matrix $M = (m_{i,j})$ by

$$m_{i,j} = \begin{cases} 1 & \text{if } \tau(i) = j, \\ 0 & \text{if } \tau(i) \neq j. \end{cases}$$

Prove that $\det M = \operatorname{sign}(\sigma)$.

5. (a) Consider the following polynomials in $\mathbb{Q}[X]$.

$$f(x) = x^5 + x^3 - x - 1, \quad g(x) = x^4 - 1.$$

Using Euclid's algorithm, find the highest common factor h of f and g. Find polynomials $a, b \in \mathbb{Q}[X]$, such that

$$h = af + bg.$$

(b) Let k be a field and let $f, g, h \in k[X]$ be three polynomials. Given that f is irreducible, show that if f is a factor of gh, then f is a factor of g or f is a factor of h.

Suppose we have irreducible monic polynomials $f_1, \ldots, f_r$ and $g_1, \ldots, g_s$ satisfying

$$f_1 \ldots f_r = g_1 \ldots g_s.$$

Prove that $r = s$ and that the g_i may be renumbered so that

$$f_1 = g_1, \quad f_2 = g_2, \; \ldots \; , \; f_r = g_r.$$

6. (a) Let V be a finite dimensional vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear map. Suppose T has only one eigenvalue $\lambda \in \mathbb{C}$;

(i) define the generalized eigenspaces $V_\lambda^{(i)}$;

(ii) explain what is meant by a *Jordan basis* of V (with respect to T).

(b) Consider the following matrix.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -6 & 3 & 2 \end{pmatrix}.$$

Find:

(i) the characteristic and minimal polynomials of A;

(ii) the Jordan canonical form of A;

(iii) a Jordan basis of $\mathbb{C}^3$ with respect to A.

END OF PAPER