UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M12A: Algebra 1

COURSE CODE : **MATHM12A**

UNIT VALUE : **0.50**

DATE : **02-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-C1037-3-200

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (i) Negate the following formula, and replace it by an equivalent one which does not involve $\neg$, $\wedge$, $\vee$ or $\exists$;

$$(\exists x)\neg P(x) \bigwedge \neg(\exists y)(\exists z)\,(Q(y) \wedge \neg R(z))\,.$$

(ii) Let σ be a permutation of the set $\{1, \ldots, n\}$; explain what is meant by saying that (a) σ is a transposition; (b) σ is an adjacent transposition.

Show that any transposition can be written as a product of odd number of adjacent transpositions.

(iii) Decompose $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 5 & 15 & 12 & 2 & 7 & 11 & 1 & 4 & 13 & 14 & 10 & 3 & 8 & 6 & 9 \end{pmatrix}$

into a product of disjoint cycles and hence compute $\mathrm{sign}(\sigma)$ and $\mathrm{ord}(\sigma)$.

2. Let $\epsilon(r, s)$ be the basic $m \times m$ matrix given by $\epsilon(r, s)_{ij} = \delta_{ri}\delta_{sj}$ where 'δ' denotes the Kronecker delta. Describe in detail the elementary $m \times m$ matrices

(i) $E(r, s; \lambda)$ $(r \neq s)$; (ii) $\Delta(r, \lambda)$ $(\lambda \neq 0)$; (iii) $P(r, s)$ $(r \neq s)$

in terms of the basic matrices $\epsilon(r, s)$.

Furthermore, show by a calculation that $E(r, s; \lambda)$ is invertible for $r \neq s$ and write down its inverse.

For the matrix A below, find A^{-1} and express A^{-1} as a product of elementary matrices; hence also express A as a product of elementary matrices.

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

3. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V; explain what is meant by saying that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

In each case below, decide with justification whether the given vectors are linearly independent. If they are not, give an explicit dependence relation between them.

(a) $$\begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix};$$

(b) $$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

Explain what is meant by a *spanning set* for a vector space V. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a spanning set for V, and suppose that $\mathbf{u} \in V$ can be expressed as a linear combination of the form

$$\mathbf{u} = \sum_{r=1}^{n} \lambda_r \mathbf{v}_r$$

with $\lambda_1 \neq 0$. Show that $\{\mathbf{u}, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is also a spanning set for V.

State and prove the Exchange Lemma.

4. Let V, W be vector spaces over a field $\mathbb{F}$ and let $T : V \to W$ be a mapping; explain what is meant by saying that T is *linear.*

When T is linear, explain what is meant by

(a) the kernel, $\mathrm{Ker}(T)$ and

(b) the image, $\mathrm{Im}(T)$.

State and prove a relationship which holds between $\dim \mathrm{Ker}(T)$ and $\dim \mathrm{Im}(T)$.

Let $T_A : \mathbb{Q}^6 \to \mathbb{Q}^4$ be the linear mapping $T_A(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \\ 4 & -8 & 3 & 1 & 0 & 3 \\ -1 & 2 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

Find (i) $\dim \mathrm{Ker}(T_A)$; (ii) a basis for $\mathrm{Ker}(T_A)$; (iii) a basis for $\mathrm{Im}(T_A)$.

5. (i) Let $f : A \to B$ be a mapping between sets A, B. Explain what is meant by saying that f is (a) injective ; (b) surjective ; (c) invertible.

In each case below decide with proof whether the given mapping is injective and also whether it is surjective.

(d) $f : \mathbb{Z} \to \mathbb{Z}$; $f(x) = 4x^3 - x$

(e) $g : \mathbb{R} \to \mathbb{R}$; $g(x) = 4x^3 - x$.

(ii) Let $\mathcal{P}_6(\mathbb{R})$ be the vector space of polynomials of degree $\leqslant 6$ over the field $\mathbb{R}$ and let $D : \mathcal{P}_6(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R})$ be the linear map given by differentiation. Write down the least positive integer n for which $D^{2n} = 0$ on $\mathcal{P}_6(\mathbb{R})$.

By factorisation of the formal expression $D^{2n} - \mathrm{I}$, or otherwise, show that the mapping

$$D^6 + D^4 + D^2 + \mathrm{I} : \mathcal{P}_6(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R})$$

is invertible, and write down an expression for its inverse in terms of D. Hence find the unique solution $\alpha \in \mathcal{P}_6(\mathbb{R})$ to the differential equation

$$\frac{d^6\alpha}{dx^6} + \frac{d^4\alpha}{dx^4} + \frac{d^2\alpha}{dx^2} + \alpha = x^6 + x^5.$$

6. Let $T : U \to V$ be a linear map between vector spaces U , V, and let $\mathcal{E} = (e_i)_{1 \leqslant i \leqslant m}$ be a basis for U and $\Phi = (\varphi_j)_{1 \leqslant j \leqslant n}$ be a basis for V. Explain what is meant by the matrix $m(T)^{\Phi}_{\mathcal{E}}$ of T taken with respect to $\mathcal{E}$ (on the left) and Φ (on the right) and prove that if $S : V \to W$ is also a linear map and $\Psi = (\psi_k)_{1 \leqslant k \leqslant p}$ is a basis for W then

$$m(S \circ T)^{\Psi}_{\mathcal{E}} = m(S)^{\Psi}_{\Phi} m(T)^{\Phi}_{\mathcal{E}}.$$

Let $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$; $\Phi = \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

be bases for $\mathbf{F}^3$ and let $T : \mathbf{F}^3 \to \mathbf{F}^3$ be the mapping

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 & +2x_2 & +2x_3 \\ -4x_1 & -x_2 & -2x_3 \\ 2x_1 & & +x_3 \end{pmatrix}.$$

Write down (i) $m(T)^{\mathcal{E}}_{\mathcal{E}}$ and (ii) $m(\mathrm{Id})^{\mathcal{E}}_{\Phi}$, and hence find $m(T)^{\Phi}_{\Phi}$.