UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *B.Sc.(Econ)M.Sci.*

Mathematics M12A: Algebra 1

COURSE CODE	:	**MATHM12A**
UNIT VALUE	:	**0.50**
DATE	:	**03-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0955-3-180

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (i) Negate the following formula, and replace the result by an equivalent formula which does not involve either $\Longrightarrow$ or $\neg$:

$$(\neg p \Longrightarrow q) \Longrightarrow \neg(\neg r \Longrightarrow s).$$

(ii) Negate the following formula, and replace it by an equivalent one which does not involve $\neg$, $\forall$, $\wedge$ or $\vee$;

$$((\forall y)(Q(x,y) \wedge \neg P(y,x))) \bigwedge (\exists x)P(x,y).$$

(iii) Prove that any cyclic permutation of the set $\{1,\ldots,n\}$ can be written as a product of adjacent transpositions.

(iv) Decompose $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 7 & 1 & 8 & 10 & 2 & 15 & 14 & 3 & 4 & 13 & 6 & 11 & 9 & 5 & 12 \end{pmatrix}$

into a product of disjoint cycles and hence compute $\text{sign}(\sigma)$ and $\text{ord}(\sigma)$.

2. Let $\epsilon(r,s)$ be the basic $m \times m$ matrix given by $\epsilon(r,s)_{ij} = \delta_{ri}\delta_{sj}$ where 'δ' denotes the Kronecker delta. Explain without proof how to calculate the product $\epsilon(r,s)\epsilon(u,t)$.

Describe in detail the elementary $m \times m$ matrices

(i) $E(r,s;\lambda)$ $(r \neq s)$; (ii) $\Delta(r,\lambda)$ $(\lambda \neq 0)$; (iii) $P(r,s)$ $(r \neq s)$

in terms of the basic matrices $\epsilon(r,s)$.

For the matrix A below, find A^{-1} and express A^{-1} as a product of elementary matrices; hence also express A as a product of elementary matrices.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

3. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V ; explain what is meant by saying that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is (i) linearly independent, and (ii) spans V.

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a spanning set for V, and suppose that $\mathbf{u} \in V$ can be expressed as a linear combination of the form

$$\mathbf{u} = \sum_{r=1}^{n} \lambda_r \mathbf{v}_r$$

with $\lambda_1 \neq 0$. Show that $\{\mathbf{u}, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is also a spanning set for V.

State the Exchange Lemma, and explain how it is used in formulating the idea of the dimension of a vector space.

In each case below, decide with justification whether the given vectors are linearly independent. If they are not, give an explicit dependence relation between them.

(a) $$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix};$$

(b) $$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix};$$

4. Let V, W be vector spaces over a field $\mathbb{F}$ and let $T : V \to W$ be a mapping; explain what is meant by saying that T is *linear* .

When T is linear, explain what is meant by (a) the kernel, $\mathrm{Ker}(T)$ and ; (b) the image, $\mathrm{Im}(T)$.

State and prove a relationship which holds between $\dim \mathrm{Ker}(T)$ and $\dim \mathrm{Im}(T)$.

Let $T_A : \mathbb{Q}^5 \to \mathbb{Q}^4$ be the linear mapping $T_A(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 2 \\ -1 & 1 & 0 & 1 & 0 \\ 2 & -2 & -1 & 0 & 1 \end{pmatrix}.$$

Find (i) $\dim \mathrm{Ker}(T_A)$; (ii) a basis for $\mathrm{Ker}(T_A)$; (iii) a basis for $\mathrm{Im}(T_A)$.

5. Let $f : A \to B$ be a mapping between sets A, B. Explain what is meant by saying that

(a) f is injective ; (b) f is surjective ; (c) f is invertible.

Prove that if f is invertible then f is both injective and surjective.

Show that the mapping $f : \mathbb{Z} \to \mathbb{Z}$; $f(x) = x^3 + x$ is not surjective, and in each case below decide with proof whether the given mapping is injective ;

(i) $g : \mathbb{R} \to \mathbb{R}$; $g(x) = x^3 + x$;

(ii) $h : \mathbb{C} \to \mathbb{C}$; $h(x) = x^3 + x$.

Let $\mathcal{P}_9(\mathbb{R})$ be the vector space of polynomials of degree $\leqslant 9$ over the field $\mathbb{R}$ and let $D : \mathcal{P}_9(\mathbb{R}) \to \mathcal{P}_9(\mathbb{R})$ be the linear map given by differentiation. Write down the least positive integer n for which $D^n = 0$ on $\mathcal{P}_9(\mathbb{R})$.

By factorisation of the formal expression $D^n - \mathrm{I}$, or otherwise, show that the mapping

$$D^5 - \mathrm{I} : \mathcal{P}_9(\mathbb{R}) \to \mathcal{P}_9(\mathbb{R})$$

is invertible, and write down

(iii) an expression for its inverse in terms of D, and

(iv) the unique solution $\alpha \in \mathcal{P}_9(\mathbb{R})$ to the differential equation

$$\frac{d^5\alpha}{dx^5} - \alpha = x^6 + x^4.$$

6. Let $T : U \to V$ be a linear map between vector space U , V, and let $\mathcal{E} = (e_i)_{1 \leqslant i \leqslant m}$ be a basis for U and $\Phi = (\varphi_j)_{1 \leqslant j \leqslant n}$ be a basis for V. Explain what is meant by the matrix $m(T)_{\mathcal{E}}^{\Phi}$ of T taken with respect to $\mathcal{E}$ (on the left) and Φ (on the right) and prove that if $S : V \to W$ is also a linear map and $\Psi = (\psi_k)_{1 \leqslant k \leqslant p}$ is a basis for W then

$$m(S \circ T)_{\mathcal{E}}^{\Psi} = m(S)_{\Phi}^{\Psi} m(T)_{\mathcal{E}}^{\Phi}.$$

Let $T : \mathbf{F}^2 \to \mathbf{F}^2$ be the mapping $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ -4x_1 - x_2. \end{pmatrix}$

and let $\mathcal{E} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ and $\Phi = \{\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$.

Write down (i) $m(T)_{\mathcal{E}}^{\mathcal{E}}$ and (ii) $m(\mathrm{Id})_{\Phi}^{\mathcal{E}}$, and hence find $m(T)_{\Phi}^{\Phi}$.

 END OF PAPER