UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M12A: Algebra 1

COURSE CODE	:	**MATHM12A**
UNIT VALUE	:	**0.50**
DATE	:	**29-APR-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0949-3-190

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let σ be a permutation of the set $\{1, \ldots, n\}$. Explain what is meant by saying (i) σ is an adjacent transposition ; (ii) σ is a cycle of length m.

   Prove that any cycle can be written as a product of adjacent transpositions.

   Define $\text{sign}(\sigma)$, and prove that if σ is a cycle of length m then

$$\text{sign}(\sigma) = (-1)^{m-1}.$$

   Decompose $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 5 & 9 & 10 & 15 & 3 & 12 & 4 & 6 & 2 & 1 & 14 & 8 & 7 & 13 & 11 \end{pmatrix}$

   into a product of disjoint cycles and hence compute

   (iii) $\text{sign}(\sigma)$ and

   (iv) $\text{ord}(\sigma)$.

2. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V ; explain what is meant by saying that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is (i) linearly independent, and (ii) spans V.

   Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for V, and let $\mathbf{u} \in V$ be such that $\mathbf{u} \neq 0$. Show that for some index i, the set $\{\mathbf{u}\} \cup \{\mathbf{v}_j : j \neq i\}$ also spans V.

   State the Exchange Lemma, and explain how it is used in formulating the idea of the dimension of a vector space.

   In each case below, decide whether the given vectors are linearly independent. If they are not, give an explicit dependence relation between them.

   (a) $\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$;

   (b) $\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

3. Let $\epsilon(r,s)$ be the basic $m \times m$ matrix

$$\epsilon(r,s)_{ij} = \begin{cases} 1 & \text{if } \; i = r \text{ and } j = s \\ 0 & \text{otherwise} \end{cases}.$$

Explain without proof how to calculate the product $\epsilon(r,s)\epsilon(u,t)$.

Describe in detail the elementary $m \times m$ matrices

(i) $E(r,s;\lambda) \quad (r \neq s)$; (ii) $\Delta(r,\lambda) \quad (\lambda \neq 0)$; (iii) $P(r,s) \quad (r \neq s)$

in terms of the basic matrices $\epsilon(r,s)$.

For the matrix

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix},$$

find A^{-1} and describe A^{-1} as a product of elementary matrices; hence also express A as a product of elementary matrices.

4. Let V be a vector space over a field $\mathbb{F}$ and let $U \subset V$; explain what is meant by saying that U is a *vector subspace* of V.

If W is also a vector space over $\mathbb{F}$, and $T : V \to W$ is a mapping, explain what is meant by saying that T is *linear* ; define

(a) the kernel, $\mathrm{Ker}(T)$; (b) the image, $\mathrm{Im}(T)$.

Prove that $\mathrm{Ker}(T)$ is a vector subspace of V.

State without proof a relationship which holds between $\dim \mathrm{Ker}(T)$ and $\dim \mathrm{Im}(T)$.

Let $T_A : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear mapping $T_A(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 1 & 3 \\ 1 & 5 & 1 & 5 \end{pmatrix}.$$

Find (i) $\dim \mathrm{Ker}(T_A)$; (ii) a basis for $\mathrm{Ker}(T_A)$; (iii) a basis for $\mathrm{Im}(T_A)$.

5. (i) Negate the following formula, and replace the result by an equivalent formula involving only p, q, r, $\Longrightarrow$, (,) ;

$$(\neg(p \wedge \neg r)) \wedge (r \wedge \neg q).$$

(ii) Negate the following formula, and replace it by an equivalent one which does not involve $\neg$, $\Longrightarrow$ or $\vee$;

$$(\exists x)(\forall y)Q(x,y) \Longrightarrow (\exists x)(\forall y)\neg P(x,y).$$

(iii) Let $f : A \to B$ be a mapping between sets A, B. Explain what is meant by saying that

(a) f is injective ; (b) f is surjective ; (c) f is invertible.

Prove that f is invertible if and only if f is both injective and surjective.

In each case below decide whether the given mapping f is (a) injective (b) surjective; moreover, if f bijective, give the explicit form of f^{-1}:

(iv) $f : \mathbb{Z} \to \mathbb{Z}$; $f(x) = 2x + 1$;

(v) $f : \mathbb{Q} \to \mathbb{Q}$; $f(x) = 2x + 1$;

(vi) $f : \mathbb{C} \to \mathbb{C}$; $f(x) = 2x^2 + 1$.

6. Explain what is meant by a *field* .

If $\mathbb{F}$ is a field and $\lambda \in \mathbb{F}$ is an element for which the equation $x^2 = \lambda$ has no solution in $\mathbb{F}$, explain in detail how the set

$$\mathbb{F}(\surd\lambda) = \{a + b\surd\lambda : \ a, b \in \mathbb{F}\}$$

may be regarded (i) as a vector space over $\mathbb{F}$, and (ii) as a field ; furthermore, derive a formula for $(a + b\surd\lambda)^{-1}$ if $a + b\surd\lambda \neq 0$.

Illustrate your remarks by displaying the multiplication table of the field $\mathbb{F}_3(\surd-1)$, where $\mathbb{F}_3$ is the field with three elements $\{0, 1, 2\}$.

Explain how to obtain the field $\mathbb{C}$ of complex numbers by this construction, and give an example of an infinite field other than $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.