UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *B.Sc.(Econ)M.Sci.*

Mathematics M12A: Algebra 1

COURSE CODE	:	**MATHM12A**
UNIT VALUE	:	**0.50**
DATE	:	**09-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0942-3-200

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Use the method of elementary row operations to find A^{-1}.

   (b) Let A and B be $n \times n$ matrices (with real entries). The product AB and tranpose A^T are defined as usual and $A * B$ is defined by $[A * B]_{ij} = [A]_{ij}[B]_{ij}$. For each of the following statements, either prove it (using i, j entries) or give a 2×2 counterexample:

   (i) $(AB)^T = A^T B^T$,

   (ii) $(A * B)^T = A^T * B^T$.

2. (a) Do a table of cases for the set operation $\triangle$ (given by $A \triangle B = (A \cap B^c) \cup (A^c \cap B)$).

   Use tables of cases to prove or disprove each of the following:

   (i) $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$,

   (ii) $A \triangle (B \cap C) = (A \triangle B) \cap (A \triangle C)$.

   (b) How is the *Cartesian product* $A \times B$ of two sets A and B defined?

   For each of the following statements either prove it or provide a counterexample:

   (i) For all sets A, B, C and D, $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$,

   (ii) For all sets A, B, C and D, $(A \times B) \triangle (C \times D) \subseteq (A \triangle C) \times (B \triangle D)$.

3. (a) Define what it means for a relation R on a set A to be (i) *reflexive*, (ii) *symmetric*, (iii) *transitive*, (iv) an *equivalence relation.* If R is an equivalence relation, define (v) *equivalence class* and (vi) a *complete set of representatives.*

   (b) For each of the following relations, state whether it is (i) reflexive, (ii) symmetric, (iii) transitive, justifying your answer. For any of the relations which are equivalence relations, give the equivalence classes and a complete set of representatives.

   (α) R defined on $\mathcal{P}(\{1, 2, ..., n\})$, the set of subsets of the set $\{1, 2, ..., n\}$, by ARB if $A \cap B \neq \emptyset$ (here n is any integer ≥ 2).

   (β) R defined on $\mathbf{Z}$ by aRb if $|b - a| \leq 2$.

   (γ) R defined on $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ by $(a, b)R(c, d)$ if there exists $(x, y) \in \mathbf{Z}^2$ such that $(c, d) = (a, b) + (x, y)$.

4. (a) State the h, k lemma. Prove that if p is a prime number and $a \not\equiv 0 \pmod{p}$, then a has an inverse $\pmod{p}$. Find $17^{-1} \pmod{53}$.
   (b) State Fermat's Little Theorem and find $17^{518} \pmod{53}$.
   (c) Prove that there are infinitely many prime numbers.

5. (a) Evaluate (i) $\sum_{r=0}^{n} {}^nC_r/2^r$, (ii) $\sum_{r=0}^{5} (-1)^r \; {}^nC_r \; {}^nC_{5-r}$.
   (b) How many functions $f : \{1, 2, 3, ..., 10\} \longrightarrow \{1, 2, 3, ..., 10\}$ are there? How many of these functions are injective? How many of these functions contain 1 in the image? How many of these functions contain both 1 and 2 in the image?
   (c) Let p, q and r be distinct primes. How many numbers between 1 and pqr (inclusive) are divisible by at least one of p, q and r?

[*Answers may be left in terms of powers, binomial coefficients, etc.*]

6. (a) Prove Lagrange's Theorem (that if G is a finite group and H a subgroup, then the order of H divides the order of G).
   (b) Show that a group of prime order must be cyclic.
   (c) Find all subgroups of S_3, explaining your reasoning.