UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. B.Sc.(Econ) M.Sci.

Mathematics M12A: Algebra 1

COURSE CODE	:	MATHM12A
UNIT VALUE	:	0.50
DATE	:	07-MAY-02
TIME	:	14.30
TIME ALLOWED	:	2 hours

02-C0937-3-170

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Find, in standard form, the complete set of solutions to the following system of equations:

$$\begin{array}{ccccccccc} x & + & y & + & z & + & w & = & 2 \\ x & + & 2y & + & 3z & + & 4w & = & 6 \\ x & & & - & z & - & 2w & = & -2 \end{array}$$

   (b) Let A and B be any $n \times n$ matrices (with real entries). For each of the following statements either prove it (using (i, j)-entries and justifying each step in your argument) or give a 2×2 counterexample:

   (i) $(A + B)C = AC + BC$,

   (ii) $(A + B)^2 = A^2 + 2AB + B^2$

2. (a) Let $f : A \longrightarrow B$ be a function. Give the definition of each of the following: (i) f is *injective*, (ii) f is *surjective*, (iii) f is *bijective*.

   (b) Let $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions, so $g \circ f : A \longrightarrow C$. For each of the following statements, either prove it or give a counterexample:

   (i) if f and g injective then so is $g \circ f$,

   (ii) if $g \circ f$ is injective, then so is f,

   (iii) if $g \circ f$ is injective, then so is g.

   (c) Prove that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 3(3x^5 - 10x^3 + 30x)^5 - 10(3x^5 - 10x^3 + 30x)^3 + 30(3x^5 - 10x^3 + 30x)$ is injective.

3. (a) The set $A - B$ is defined as $\{x \in A : x \notin B\}$. Use tables of cases to prove or disprove the following:

   (i) $A - (B \cup C) = (A - B) \cap (A - C)$,

   (ii) $A - (B - C) = (A - B) \cup C$.

   (b) How is the *Cartesian product* $A \times B$ of two sets A and B defined? For each of the following statements, either prove it or give a counter-example:

   (i) $A \times (B - C) \subseteq (A \times B) - (A \times C)$,

   (ii) $(A \times A) - (B \times B) \subseteq (A - B) \times (A - B)$.

4. (a) Prove that if p is a prime and $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \pmod{p}$ (Fermat's Little Theorem).

   (b) Find $2^{579} \pmod{59}$.

   (c) Show that for any positive integers a and b, $ab(a^{12} - b^{12})$ is divisible by 210.

5. (a) Evaluate (i) $\sum_{r=0}^{n}(-2)^r \, {}^nC_r$, (ii) $\sum_{r=0}^{n-1} ({}^nC_{n-r})({}^nC_{r+1})$.

   (b) How many functions $f : \{1,2,3\} \longrightarrow \{1,2,3,4,5\}$ are there? How many of these functions are injective? For how many of these functions does the image consist of exactly two elements?

   (c) Let X be the set of integers between 1 and 999 inclusive. How many numbers in X have at least one digit 9 in them? How many numbers in X are divisible by at least one of 2,3 and 5?

   [*Answers may be left in terms of binomial coefficients, powers, etc.*]

6. (a) What conditions must a subset H of a group G satisfy to be a *subgroup*? Determine which of the following sets H are subgroups of $GL_2(\mathbb{R})$ (the group of 2×2 invertible real matrices under multiplication). Justify your answers.

   (i) $H = \{A \in G : A^T = A\}$, (ii) $H = \{A \in G : A^T = A^{-1}\}$.

   (b) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 6 & 7 & 4 & 5 & 1 & 8 & 9 & 3 \end{pmatrix}$, $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 4 & 3 & 6 & 7 & 8 & 5 & 9 \end{pmatrix}$ be two elements of S_9.

   (i) Express σ as a product of disjoint cycles and also as a product of transpositions. Is $\sigma \in A_9$?

   (ii) Find $\mu\sigma\mu^{-1}$ as a product of disjoint cycles. What is its order?