Exam Questions for B12

Argument and Formal Reasoning

May 1998

Answer THREE questions.

If you answer MORE THAN THREE questions only the best three will be counted.

1. a. State the conditions that have to be satisfied for a proof by induction and a proof by strong induction.

[11 marks]

b. A t-*tree* is defined to be one where each internal node has exactly t branches below it. Prove the following two results for a t-tree with n internal and s external nodes by induction and also by strong induction:

i.

$$s = (t-1).n + 1$$

[11 marks]

ii. if the internal path length is $I(n)$ and the external path length is $E(n)$ then

$$E(n) = (t-1).I(n) + t.n$$

[11 marks]

[Total = 33 marks]

2. a. Define countability of a set.

[5 marks]

b. Give two examples of a countable set and of an uncountable set, indicating why each of these sets have the property claimed for them.

[4 marks]

c. Prove that the set of all finite strings of symbols chosen from a finite alphabet of symbols is countable.

[7 marks]

d. What happens if the symbols are chosen from an infinite but countable alphabet?

[4 marks]

e. Explain briefly the notion of a Turing machine.

[8 marks]

f. Are there functions from $\mathbb{N}$ to $\mathbb{N}$ which cannot be computed by a Turing machine? Give reasons for your answer.

[5 marks]

[Total = 33 marks]

3. Let the Boolean operator $\rightarrow$ have semantics defined by the truth table

		b	
	$\rightarrow$	0	1
a	0	1	1
	1	0	1

a. Show that

i. $(p.(p \rightarrow q)) \rightarrow q$ ('modus ponens')

[5 marks]

ii. $(\bar{q}.(p \rightarrow q)) \rightarrow \bar{p}$ ('modus tollens')

[5 marks]

b. Using the identities for boolean algebra (but not using a truth table) prove that

$$((p \rightarrow q).(q \rightarrow r)) \rightarrow (p \rightarrow r)$$

[7 marks]

c. Give the inference rules for the connectives $\rightarrow, \neg, \wedge, \vee$ of propositional logic and explain briefly how they are used.

[7 marks]

d. Using the appropriate inference rules prove that

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Explain all steps in your proof.

[9 marks]

[Total = 33 marks]

4. This question is about propositional logic.

   a. Find equivalents to each of the formulas below in a propositional language using negation ($\neg$) and implication ($\rightarrow$) only.

      1. $(p \vee q)$
      2. $(p \wedge q)$
      3. $(p \wedge \neg(q \vee \neg r))$

   [9 marks]

   b. Let $\perp$ be a propositional constant that always evaluates to false. Find equivalents to each of the formulas below in a propositional language using implication ($\rightarrow$) and $\perp$ only.

      1. $\neg p$
      2. $(p \wedge q)$
      3. $(p \wedge \neg(q \vee \neg r))$

   [9 marks]

   c. What is meant by *disjunctive normal form*?

   [3 marks]

   d. For each of the formulas below construct a complete tableau with the formula at the root. Use the tableau to find an equivalent formula written in disjunctive normal form.

      1. $\neg((p \rightarrow q) \rightarrow p)$
      2. $\neg((\neg p \vee q) \rightarrow (\neg p \wedge q))$
      3. $\neg(p \rightarrow (q \rightarrow p))$
      4. $((p \rightarrow (q \vee \neg r)) \wedge (q \rightarrow (r \vee p)))$

   [12 marks]

[Total 33 marks]

5. This question is about predicate logic. Let L be a first-order language with predicate symbols $\{O^2, S^2, M^2, Eq^2, F^1\}$, no function symbols, and constant symbol $\{j\}$. Let $M = (D, I)$ be an L-structure where D is some set of people and I interprets the symbols as follows.

   - $I(O^2)(a, b)$ means that a is older than b.
   - $I(S^2)(a, b)$ means that a is the sibling (either brother or sister) of b.
   - $I(M^2)(a, b)$ means that a is the mother of b.
   - $I(Eq^2)(a, b)$ means that a is equal to b.
   - $I(F^1)(a)$ means that a is female.
   - $I(j) =$ Jane (where Jane is some member of the set D).

   a. Using this language translate the following English sentences into predicate logic.

      1. Jane is an only child.
      2. Everyone has a mother, but only one mother.
      3. Any two siblings have the same mother.
      4. Not everyone has the same mother.

   [16 marks]

   b. Let D consists of four people: Jane, David, Kambiz and Soraya. Let Jane be younger than David who is younger than Kambiz who is younger than Soraya. Let Kambiz be the brother of Soraya and let Jane be the sister of David (and there are no more siblings). Let Jane and Soraya be the only females and suppose that Soraya is the mother of David and Jane. Which of the following formulas are valid in this structure $M = (D, I)$? Explain your answers briefly.

      1. $\exists x \forall y (O^2(x, y) \vee Eq^2(x, y))$
      2. $O^2(x, j)$
      3. $\forall x \exists y (F^1(y) \wedge (Eq^2(x, y) \vee S^2(x, y)))$
      4. $(S^2(x, y) \leftrightarrow S^2(y, x))$.

   [17 marks]

   [Total 33 marks]

END OF PAPER