*Answer **THREE** questions.*

If you answer MORE THAN THREE questions only the best three will be counted.

1. *Define* the Boolean operators 'equivalent', $\equiv$, and 'not equivalent', $\not\equiv$. *Show* both operators are associative. What are the values of $a \equiv 0$ and $a \not\equiv 1$ (where 0 and 1 stand for *false* and *true* respectively)?

[10 marks]

A 'Gray Code' is defined to be an ordered sequence of binary patterns, each of N bits, where only one bit changes in going from one pattern to the next in the sequence. For instance, the patterns 000, 001, 011, 010, 110, 111, 101, 100 form a 3 bit Gray Code. If the binary integers, in numerical order, of not more than N bits are represented by successive patterns $b_1b_2\ldots b_N$, with b_1 as the most significant digit, show that the patterns $g_1g_2\ldots g_N$ will form a Gray Code if the g_i are defined by

$$g_1 = b_1 \; ; \; g_{k+1} = b_{k+1} \not\equiv b_k \text{ for } k = 1 \text{ to } N - 1 \; .$$

[10 marks]

If we are given the Gray Code defined above, *show* that the binary integers can be recovered by the operations

$$b_1 = g_1 \; ; \; b_{k+1} = g_{k+1} \not\equiv g_k \not\equiv \ldots \not\equiv g_1 \; .$$

[10 marks]

Is a code where 01010110 and 01110110 are adjacent members an acceptable Gray Code for the integers in the range 0 to 255 ?

[3 marks]

[Total 33 marks]

2. This question is about propositional logic.

a) Let *p* stand for 'John eats too much', *q* stand for 'John takes his medicine' and *r* stand for 'John is healthy'. Translate the following English sentences into propositional logic:

i) 'John is ill';

ii) 'If John does not take his medicine then he will be ill';

iii) 'Whenever John overeats he gets ill, unless he takes his medicine'.

[9 marks]

[QUESTION 2 continues over page]

[QUESTION 2 continued]

b) Find equivalents to each of the formulas below in a propositional language using only negation ($\neg$) and conjunction ($\wedge$):

i) $(p \to q)$

ii) $\neg((p \vee q) \to r)$

iii) $(p \to \neg(p \wedge \neg q))$

[10 marks]

c) For each of the formulas below construct a complete tableau for the negated formula and use this to determine whether the formula is valid or not. If the formula is not valid give an assignment to the propositions that satisfies the formula.

i) $(p \to (q \to p))$

ii) $((\neg p \vee \neg(p \to \neg q)) \wedge (\neg q \vee \neg(p \to \neg q)))$

14 marks]

[Total 33 marks]

3. This question is about predicate logic.

a) Let L be a first-order language, $M = (D, I)$ be an L-structure and let ϕ be a formula of L.

i) Say what it means for ϕ to be *valid* in M.

ii) Say what it means for ϕ to be *satisfiable* in M.

[9 marks]

b) Let L be a first-order language with constant symbol $C = \{z\}$, function symbol $F = \{add^2\}$ and predicate symbols $P = \{eq^2, g^2\}$. Let Z be the structure $\{\mathbf{Z}, I\}$ where $\mathbf{Z}$ is the set of integers, $I = (I_c, I_f, I_p)$ and

$$\begin{aligned} I_c(z) &= 0 \\ I_f(add^2) &: (m, n) \mapsto m + n \\ I_p(eq^2) &= \{(n, n) : n \in \mathbf{Z}\} \\ I_p(g^2) &= \{(m, n) : m > n\} \end{aligned}$$

For each formula below say if the formula is valid in Z and if it is satisfiable in Z. If the formula is not valid, give an assignment which falsifies the formula. If the assignment is satisfiable, give an assignment that satisfies it.

[QUESTION 3 continues on next page]

[QUESTION 3 continued]

(i) $\neg((g^2(x, y) \vee g^2(y, x)) \vee eq^2(x, y))$

(ii) $\forall x \forall y((g^2(x, y) \vee g^2(y, x)) \vee eq^2(x, y))$

(iii) $(g^2(x, y) \rightarrow \exists z(g^2(x, z) \wedge g^2(z, y)))$

(iv) $(g^2(x, 0) \wedge \forall y(g^2(y, 0) \rightarrow (g^2(y, x) \vee eq^2(y, x))))$.

[24 marks]

[Total 33 marks]

4. What is meant by *a proof by induction*? Give an exact definition.

Given that the interior angles of a triangle add up to π radians (2 right angles) *prove by induction* that the interior angles of a polygon with n sides add up to $(n - 2)\pi$ radians.

[11 marks]

What is meant by a *proof by strong induction*? How does this differ from a proof by induction?

Use *strong induction to prove* that the interior angles of a polygon with n sides add up to $(n - 2)\pi$ radians.

[11 marks]

A student tries to prove the proposition '*all horses have the same colour*' by using induction. The proof goes as follows:

> 'Clearly the proposition is true if there no horses and also if there is 1 horse. So, we have a base case and can move to the next case. Now assume that the proposition is true for a set of N horses. Remove one of these horses, leaving N - 1 horses of the same colour. Bring another horse from somewhere. We now have a set of N horses, which by assumption have the same colour. Bring back the horse that was sent away (which had the same colour as those that had been kept). We thus have a set of N + 1 horses which have the same colour, proving our assertion.'

As horses are known to have different colours, what is wrong with this alleged proof?

[11 marks]

[Total 33 marks]

5. What is meant by the statement '*the infinite set A is countable*'? Why is the notion of countability important?

[10 marks]

If a_1, a_2, a_3, … are a countable set of characters, possibly an infinite set, *prove* that the set of all words consisting of a finite set of characters is countable.

[10 marks]

D is the set of all sets of positive integers that can be defined by English sentences of finite length. For instance, the set {0, 2, 4, 6, …} is defined by the sentence 'these are the even integers' and therefore belongs to D.

Prove that D is an infinite, countable set.

[10 marks]

The set D was defined in terms of 'English sentences': does the choice of language affect your solution?

[3 marks]

[Total 33 marks]

END OF PAPER